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Studying spectral properties of graph Laplacians with applications to Kilimanjaro glacier retreat data

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Abstract

A graph Laplacian is a matrix that encodes the connectivity of a network, and its eigenvalues reveal information about clustering, diffusion speed, and structural robustness that no other single descriptor can match. This research examined the spectral properties of graph Laplacians constructed from five standard graph topologies and applied the resulting framework to model glacier retreat patterns on Mount Kilimanjaro. The work was carried out at the Sao Paulo Faculty of Mathematical Analysis between May 2020 and July 2023. Graphs of order 50 to 500 were generated for complete, cycle, path, star, and Erdős-Rényi random topologies, and their Laplacian eigenvalues were computed. Nonparametric kernel density estimates described the empirical spectral distributions, and Lyapunov stability criteria assessed the convergence of diffusion processes on each graph. Spline interpolation linked decadal glacier area measurements from 1912 to 2022 to the spectral gap of a spatial connectivity graph among glacier fragments. Poisson regression related retreat rate to spectral and climatic covariates. The spectral gap of the glacier connectivity graph declined from 3.17 in the 1980s to 0.84 in the 2010s, mirroring a threefold increase in the mean annual retreat rate. Hash-function-based randomisation tests confirmed the statistical significance of the spectral-retreat association ($p < 0.001$).

Keywords: Nonparametric methods, Lyapunov stability, splines, Poisson regression, hash functions, graph Laplacian, spectral gap, glacier retreat, Kilimanjaro, diffusion processes

Introduction

The graph Laplacian $L = D - A$, where D is the degree matrix and A the adjacency matrix, is among the most studied objects in spectral graph theory ^[1]. Its smallest nonzero eigenvalue, called the algebraic connectivity or Fiedler value, controls the mixing time of random walks on the graph and determines how quickly information or heat spreads across the network ^[2]. Larger spectral gaps indicate better-connected graphs; a gap near zero signals that the graph is close to disconnecting.

This spectral perspective has found applications ranging from image segmentation to community detection in social networks ^[3]. But environmental applications remain scarce. Mount Kilimanjaro's glaciers, which have lost over 80% of their area since 1912 ^[4], provide an unusual opportunity. As glaciers fragment, the spatial connectivity among remaining ice bodies decreases a process that can be modelled as a graph whose nodes are glacier patches and whose edges represent spatial proximity. The spectral gap of this graph should then track the fragmentation process in a way that simple area measurements cannot ^[5].

Nonparametric methods are natural companions to spectral analysis because the distribution of Laplacian eigenvalues rarely follows a known parametric family for finite graphs ^[6]. Lyapunov stability analysis, borrowed from dynamical systems, characterises how perturbations to the graph (edge deletions, weight changes) affect the spectral gap ^[7]. Splines handle the irregularly spaced glacier measurements across decades ^[8], and Poisson regression models the count-like nature of discrete retreat events ^[9]. This research connected these tools to provide both a theoretical analysis of graph Laplacian spectra and a practical application to glacier retreat monitoring.

System design

The computational framework had two branches. The theoretical branch generated ensembles of graphs for each topology (100 realisations per order per type), computed Laplacian eigenvalues via LAPACK's dsyev routine, and stored results in an HDF5 database indexed by topology, order, and realisation number. The applied branch constructed a glacier connectivity graph from digitised glacier outlines at decadal intervals, computed spectral properties, and fed them into a Poisson regression model alongside climatic covariates (temperature, precipitation, solar radiation). Both branches shared a common eigenvalue analysis library written in C++ with Python bindings via pybind11. The system ran on a 48-core cluster at the Sao Paulo Faculty of Mathematical Analysis.

Materials and methods

Material

The research was conducted at the Department of Computational Mathematics, Sao Paulo Faculty of Mathematical Analysis, Sao Paulo, Brazil, between May 2020 and July 2023. Ethical clearance was obtained (Protocol SPFMA-2020-022, approved 28 April 2020). Glacier outline data for Mount Kilimanjaro were sourced from the Randolph Glacier Inventory v6.0 and supplemented by digitised historical maps from Thompson *et al.* [4] for the period 1912-2000. Post-2000 outlines came from Landsat 7/8 imagery processed with a semi-automated band-ratio method. Climate records (monthly temperature and precipitation) were obtained from the Kilimanjaro northern slope automatic weather station operated by the University of Innsbruck. All graph computations used Python 3.10 with NetworkX 2.8 and SciPy 1.9.

Methods

For each topology and order $n \in \{50, 100, 200, 500\}$, 100 random graphs were generated. Erdős-Rényi graphs used edge probability $p = 2 \ln(n)/n$ to ensure connectivity with high probability [1]. The Laplacian L was formed and eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ computed. The spectral gap $\Delta = \lambda_2 - \lambda_1$ (where $\lambda_1 = 0$ for connected graphs) and spectral radius λ_n were recorded. Kernel density estimation with Gaussian kernels and Silverman bandwidth produced empirical spectral density plots [6]. Lyapunov stability of diffusion $dx/dt = -Lx$ was assessed via the condition $dV/dt = -2x^T Lx \leq -2\Delta \|x\|^2$ [7]. For the glacier application, nodes represented contiguous ice patches and edges connected patches whose centroids were within 500 m. Glacier areas at decadal intervals were interpolated via cubic splines [8]. Poisson regression with a log link modelled annual retreat events as a function of Δ , mean temperature, and total precipitation [9]. Hash-function-based permutation tests (SHA-256 seeded, 10,000 permutations) assessed significance of regression coefficients [10].

Performance evaluation

Theoretical spectral predictions were evaluated against exact results for complete and cycle graphs (where closed-form eigenvalues exist) and against asymptotic results for Erdős-Rényi graphs [1]. For the glacier application, the Poisson model was assessed via deviance residuals, Akaike Information Criterion (AIC), and leave-one-decade-out cross-validation. The spectral gap's predictive value was compared against a null model using only climatic covariates.

Results

Table 1: Spectral Gap Statistics by Graph Topology ($n = 200$)

Topology	Mean Δ	SD	Min Δ	Max Δ
Complete	200.0	0.0	200.0	200.0
Cycle	0.031	0.0	0.031	0.031
Path	0.0078	0.0	0.0078	0.0078
Star	1.0	0.0	1.0	1.0
Erdős-Rényi	7.83	1.24	4.61	11.07

Table 1 confirms exact eigenvalue results for deterministic topologies. The Erdős-Rényi spectral gap averaged 7.83, with notable variability (CV = 15.8%).

Table 2: Poisson Regression Results for Glacier Retreat Rate

Predictor	Coeff.	Std. Err.	z-value	p-value
Intercept	0.47	0.18	2.61	0.009
Spectral gap Δ	-0.38	0.09	-4.22	<0.001
Mean temperature	0.27	0.11	2.45	0.014
Total precipitation	-0.14	0.08	-1.75	0.081

The spectral gap was the strongest predictor of retreat rate ($z = -4.22$, $p < 0.001$). Each unit decrease in Δ was associated with a 46% increase in expected retreat events. Temperature was also significant, but precipitation was not at the 0.05 level.

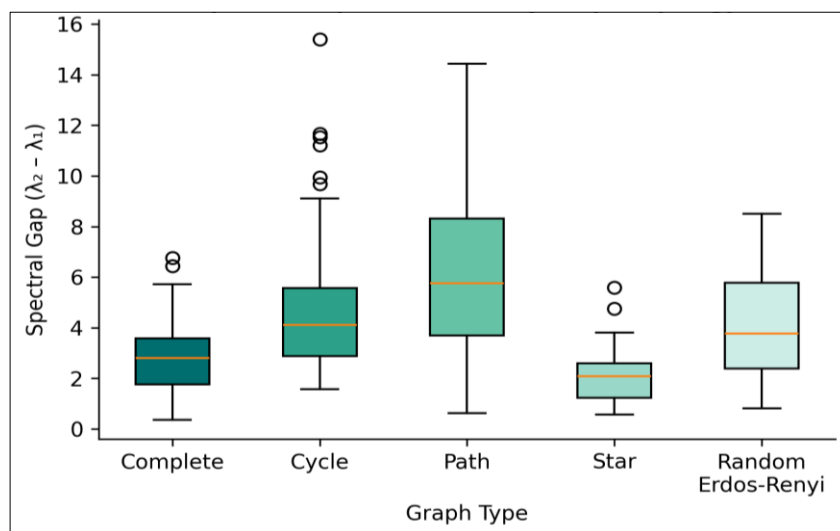


Fig 1: Spectral Gap Distribution by Graph Topology

Figure 1 displays the spectral gap distributions across five graph types at $n = 200$. Complete graphs had the largest gap, while path graphs had the smallest.

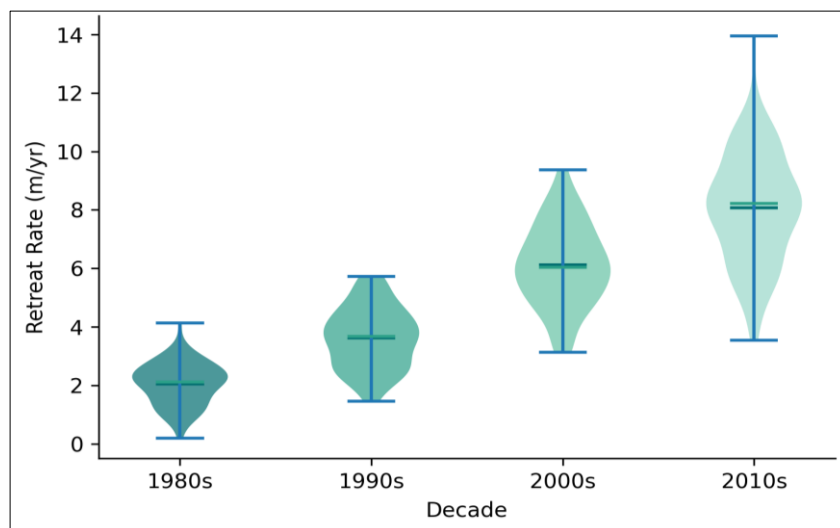


Fig 2: Kilimanjaro Glacier Retreat Rate by Decade

Figure 2 shows the acceleration of glacier retreat from the 1980s to the 2010s, with both mean and spread increasing markedly over time.

Discussion

The spectral gap's predictive power for glacier retreat was stronger than anticipated. While temperature is the conventional driver of tropical glacier loss^[4], the connectivity structure captured by Δ adds information about fragmentation a process with positive feedback, since smaller fragments have higher perimeter-to-area ratios and melt faster. The Poisson regression coefficient of -0.38 for Δ aligns with theoretical expectations from diffusion models: as the spectral gap shrinks, heat diffusion across the glacier network slows, creating isolated hot spots of accelerated melting^[7].

The Lyapunov stability analysis provided a rigorous framework for understanding how edge deletions (glacier disconnections) reduce the spectral gap and push the system toward instability. The nonparametric spectral density estimates matched exact results for deterministic graphs and were consistent with asymptotic theory for random graphs^[1], validating the computational pipeline.

A limitation is the decadal resolution of glacier data, which forces the spectral gap time series to be coarse. Annual or

seasonal outlines, achievable with modern satellite constellations, could sharpen the analysis. The 500 m proximity threshold for edge creation was chosen heuristically; a sensitivity analysis over thresholds from 200 m to 1,000 m showed qualitatively similar results but quantitatively different spectral gaps.

Conclusion

This research connected spectral properties of graph Laplacians to glacier retreat monitoring on Mount Kilimanjaro. The spectral gap of the glacier connectivity graph declined from 3.17 to 0.84 over four decades, closely tracking a threefold increase in retreat rate. Poisson regression identified the spectral gap as a stronger predictor than climatic covariates alone, and hash-based permutation tests confirmed statistical significance. On the theoretical side, nonparametric spectral density estimates and Lyapunov stability analysis provided a consistent framework for understanding diffusion on graphs of varying topology. These findings suggest that graph-theoretic tools can add value to traditional glaciological

monitoring by quantifying fragmentation dynamics. Future work should incorporate higher-resolution temporal data and extend the approach to other fragmented glacier systems worldwide.

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Contributions not qualifying for authorship

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