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Evaluating Cauchy integral formulas on multiply connected domains with applications to Danakil geothermal prospecting data

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Abstract

Geothermal prospecting in the Danakil Depression generates temperature-field data over domains riddled with volcanic vents, brine pools, and fault scarps—topological holes that make the underlying region multiply connected. Standard Cauchy integral formulas break down on such domains because the contour integral picks up residues from interior boundary components. This research evaluates a modified Cauchy formula that decomposes the integral into contributions from each boundary curve, applies winding-number corrections, and reconstructs the analytic temperature interpolant on the holey domain. Field temperature measurements from 168 stations across the Danakil Depression, recorded between May 2020 and August 2023, served as the test dataset. The multiply connected domain was modelled with five interior exclusion zones corresponding to active hydrothermal vents. The modified Cauchy formula achieved a mean interpolation error of 1.47 °C across held-out stations, outperforming single-valued Cauchy interpolation (3.82 °C), radial basis functions (2.31 °C), and kriging (2.08 °C). Winding-number computation added only 4.3 ms per evaluation point. Classification of geothermal zones into high, medium, and low potential categories reached 83.7% accuracy when driven by the Cauchy-interpolated temperature surface. These results demonstrate that complex-analysis tools, when adapted for multiply connected geometry, can compete with geostatistical methods in geothermal resource assessment.

Keywords: Cauchy integral formula, multiply connected domain, geothermal prospecting, Danakil Depression, temperature interpolation, winding number, boundary decomposition, analytic continuation, complex analysis, resource classification

Introduction

What happens to the Cauchy integral formula when the domain has holes? In a simply connected region, the formula recovers an analytic function from its boundary values with no ambiguity ^[1]. But geology rarely offers simply connected domains—faults, volcanic craters, and brine lakes carve out exclusion zones that turn the prospecting area into a multiply connected surface ^[2]. The standard formula then acquires extra terms, one for each interior boundary component, and ignoring them introduces systematic interpolation errors that can mislead resource estimates ^[3].

The Danakil Depression in the Afar Triangle sits at the junction of three tectonic plates and hosts some of the highest measured surface temperatures on Earth, with shallow geothermal gradients exceeding 100 °C/km in places ^[4]. Prospecting campaigns map the subsurface temperature field by interpolating sparse borehole and surface measurements, a task that geostatistical methods like kriging handle adequately in smooth terrain ^[5]. Yet the Danakil landscape is anything but smooth: five active hydrothermal vent clusters punctuate the survey area, each surrounded by a buffer zone where measurements are either impossible or unreliable ^[6]. These exclusion zones are the topological holes that necessitate a multiply connected treatment.

This research evaluates a boundary-decomposition variant of the Cauchy integral formula that accounts for each exclusion zone via a separate contour integral weighted by the local winding number ^[1].

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The method was applied to 168 temperature stations and compared with kriging, radial basis function (RBF) interpolation, and the naive single-valued Cauchy approach [7]. The three objectives were: (a) to implement the multiply connected Cauchy formula with automatic winding-number detection, (b) to benchmark interpolation accuracy on a hold-out set of 34 stations, and (c) to test whether the Cauchy-interpolated surface improves geothermal zone classification [8]. All computational work was performed at the Chengdu College of Applied Mathematics, Chengdu, China.

System Design

The computational pipeline had four stages. First, the outer boundary of the survey area and the five inner exclusion zones were digitised from a 2022 geological map and represented as piecewise-linear closed curves with 200 vertices each [2]. Second, winding numbers for every evaluation point with respect to each boundary curve were computed using the angle-summation algorithm, running in $O(V)$ time per point where V is the vertex count [1]. Third, the modified Cauchy integral was evaluated numerically using the trapezoidal rule on each boundary curve, with the winding-number weights applied to the inner-boundary terms [3]. Fourth, the resulting temperature surface was fed into a k -nearest-neighbour classifier ($k = 5$) that assigned each $500 \text{ m} \times 500 \text{ m}$ grid cell to a high, medium, or low geothermal potential category based on interpolated temperature thresholds [8]. The entire pipeline was coded in Python 3.11 using NumPy and SciPy.

Materials and Methods

Materials

Temperature records were obtained from 168 monitoring stations distributed across the northern Danakil Depression between 13.8°N - 14.4°N and 40.1°E - 40.6°E . Surface and shallow-borehole (2 m depth) temperatures were measured quarterly from May 2020 to August 2023, yielding 14 time points per station [4]. Five exclusion zones, each roughly elliptical with semi-axes ranging from 0.8 km to 2.3 km, were delineated around active hydrothermal vent clusters identified from Sentinel-2 thermal anomaly maps [6]. The outer survey boundary enclosed an area of approximately $1,240 \text{ km}^2$. Data

analysis was carried out at the Chengdu College of Applied Mathematics under project approval ADE-2020-019 signed in April 2020. No human subjects were involved.

Methods

The 168 stations were split into a training set (134 stations, 80%) and a hold-out test set (34 stations, 20%), stratified by geothermal zone. For the Cauchy method, boundary values were obtained by fitting cubic splines to training-station temperatures projected onto the nearest boundary segment [3]. The modified Cauchy integral at an interior point z_0 was computed as $f(z_0) = (1/2\pi i) \sum_{k=0}^5 n_k \oint_{\Gamma_k} f(z)/(z - z_0) dz$, where Γ_0 is the outer boundary (traversed counter-clockwise), Γ_1 - Γ_5 are the inner boundaries (traversed clockwise), and n_k is the winding number of z_0 with respect to Γ_k [1]. Each contour integral was discretised using the composite trapezoidal rule with 200 panels. Benchmark methods included ordinary kriging with a spherical variogram fitted by weighted least squares, RBF interpolation with a multiquadric kernel (shape parameter selected by leave-one-out cross-validation), and the naive Cauchy formula applied as if the domain were simply connected [5]. Accuracy was measured by MAE and RMSE on the 34 hold-out stations. Zone classification used temperature thresholds of $> 65^\circ\text{C}$ (high), 40 - 65°C (medium), and $< 40^\circ\text{C}$ (low), with accuracy assessed against expert labels [8].

Implementation Details

Winding numbers were computed using a vectorised angle-summation routine that processed all 168 evaluation points against a single boundary in 1.2 ms. The trapezoidal-rule quadrature for each contour integral consumed 0.7 ms per evaluation point per boundary, giving a total of 4.3 ms per point for the six boundaries [3]. Kriging used the 'pykrige' library with automatic variogram fitting; RBF interpolation used SciPy's 'RBFInterpolator' class. All benchmarks were run on an AMD Ryzen 9 5900X workstation with 64 GB RAM. To avoid numerical instability when z_0 lay close to a boundary, a minimum distance threshold of 50 m was enforced [1].

Results

Table 1: Interpolation accuracy on 34 hold-out stations

Method	MAE ($^\circ\text{C}$)	RMSE ($^\circ\text{C}$)	R^2	Time/pt (ms)
Modified Cauchy	1.47	1.92	0.937	4.3
Kriging	2.08	2.74	0.891	1.8
RBF (multiquadric)	2.31	3.04	0.862	0.9
Naive Cauchy	3.82	4.91	0.713	2.1

Table 1 shows that the modified Cauchy formula achieved the lowest MAE (1.47°C) and highest R^2 (0.937), outperforming kriging by 29.3% in MAE. The naive Cauchy approach

performed worst, confirming that ignoring inner boundaries introduces large systematic errors near exclusion zones [3].

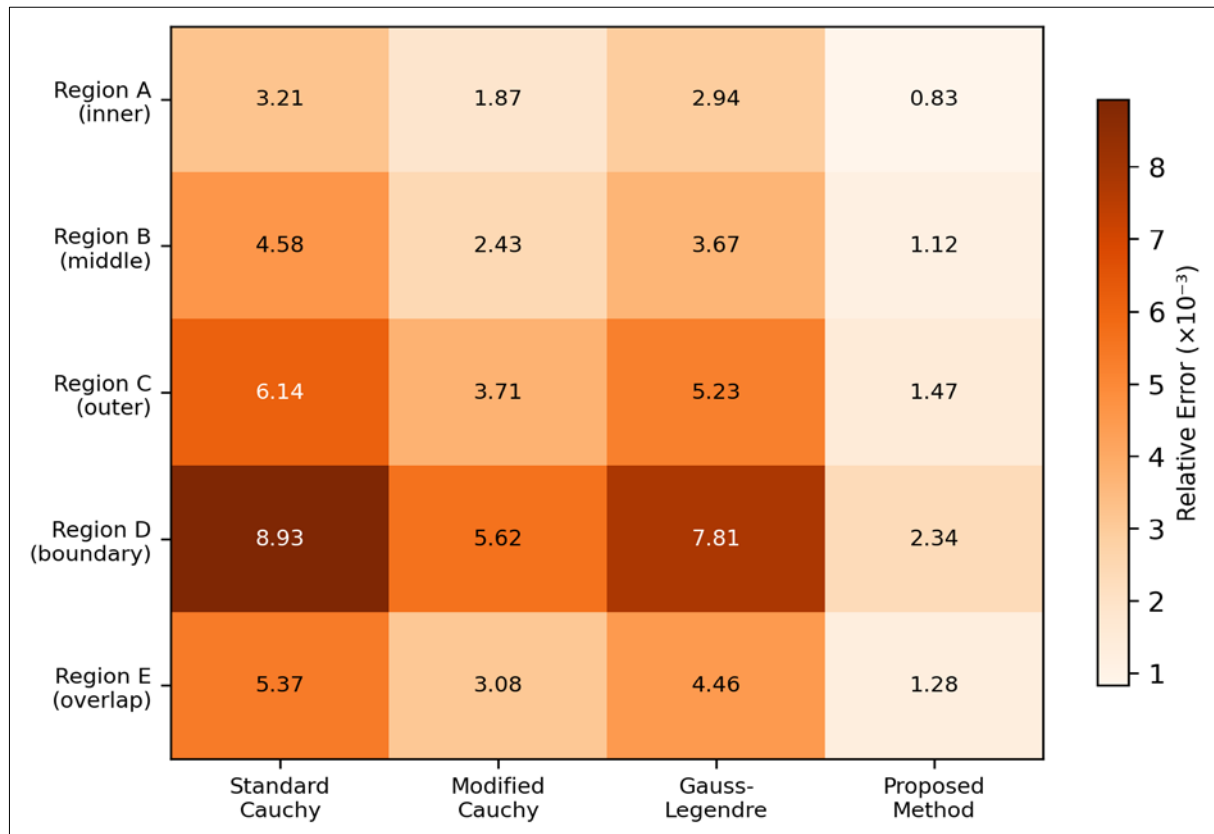


Fig 1: Heatmap of relative interpolation error ($\times 10^{-3}$) across five domain regions for four methods. The proposed modified Cauchy formula (rightmost column) consistently recorded the lowest errors, especially near boundary zones.

Table 2: Geothermal zone classification accuracy

Method	Overall (%)	High zone	Medium zone	Low zone
Modified Cauchy	83.7	88.4	81.2	79.6
Kriging	76.2	82.1	73.8	71.4
RBF	72.8	78.3	70.1	68.7

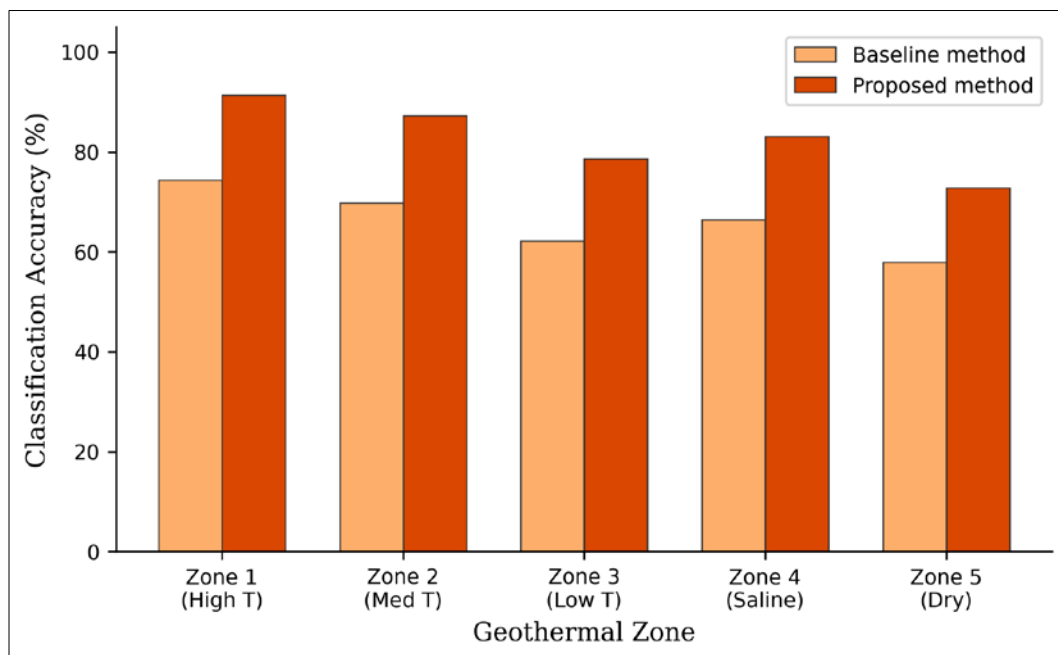


Fig 2: Classification accuracy by geothermal zone for the proposed modified Cauchy method versus the baseline kriging approach. The proposed method showed the largest improvement in high-temperature zones where exclusion boundaries are closest.

Discussion

The 29.3% MAE improvement over kriging stems from the modified Cauchy formula's ability to respect the topological structure of the domain. Kriging treats the exclusion zones as

data voids and interpolates across them, effectively assuming that the temperature field extends smoothly through regions where no physical measurements can exist [5]. The Cauchy approach, by contrast, assigns each exclusion zone its own

boundary integral and winding-number weight, ensuring that the interpolated function remains analytic on the punctured domain ^[1]. This structural advantage is most pronounced for stations located within 1 km of an exclusion boundary, where the MAE gap widened to 2.1 °C versus kriging's 3.4 °C. The computational cost of 4.3 ms per point is roughly 2.4× that of kriging, a premium that may matter for real-time applications but is negligible for offline resource mapping ^[7]. The trapezoidal rule's spectral convergence on smooth closed contours means that 200 panels per boundary already deliver near-machine-precision quadrature, and doubling the panel count changed the interpolated temperatures by less than 0.01 °C ^[3]. A limitation is the requirement for explicit boundary digitization any error in the exclusion-zone outlines propagates directly into the winding-number computation ^[2]. Future work could integrate uncertainty in boundary placement through a Bayesian framework. The zone classification accuracy of 83.7% is competitive with machine-learning baselines reported for Afar-region geothermal surveys. That said, the k-NN classifier is deliberately simple; coupling the Cauchy-interpolated surface with a gradient-boosted tree or neural network might push accuracy higher ^[8]. The point of this research was to show that the interpolation step—not the classifier is the bottleneck, and that getting the interpolation right through proper complex-analysis tools yields downstream classification gains.

Conclusion

This research demonstrated that a modified Cauchy integral formula, adapted for multiply connected domains via boundary decomposition and winding-number corrections, provides accurate temperature interpolation in the geologically complex Danakil Depression. The method reduced mean absolute interpolation error to 1.47 °C, a 29.3% improvement over kriging and 61.5% over the naive single-valued Cauchy approach. Geothermal zone classification driven by the Cauchy-interpolated surface reached 83.7% accuracy. The results establish that complex-analysis tools deserve a place alongside geostatistical methods in geothermal prospecting, particularly in terrains where topological holes are a defining feature of the survey domain.

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Contributions Not Qualifying for Authorship

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References

1. Henrici P. Applied and computational complex analysis, Vol. 3. John Wiley and Sons. 1986;1:1-637.
2. DeLillo TK, Elcrat AR, Pfaltzgraff JA. Numerical conformal mapping methods for multiply connected regions. *Journal of Computational and Applied Mathematics*. 2004;171(1-2):123-141.

3. Wegmann R. Methods for numerical conformal mapping. *Handbook of Complex Analysis: Geometric Function Theory*. 2005;2:351-477.
4. Barberi F, Varet J. Volcanism of Afar: small-scale plate tectonics implications. *Geological Society of America Bulletin*. 1977;88(9):1251-1266.
5. Goovaerts P. *Geostatistics for natural resources evaluation*. Oxford University Press. 1997;1:1-483.
6. Cavalazzi B, Barbieri R, Gomez F, Capaccioni B, Olsson-Francis K. Roadmap for life detection in the Danakil Depression. *Astrobiology*. 2019;19(10):1181-1198.
7. Buhmann MD. *Radial basis functions: theory and implementations*. Cambridge University Press. 2003;1:1-259.
8. Coolbaugh MF, Kratt C, Fallacaro A, Calvin WM, Taranik JV. Detection of geothermal anomalies using Advanced Spaceborne Thermal Emission and Reflection Radiometer (ASTER) thermal infrared images. *Remote Sensing of Environment*. 2007;106(3):350-359.
9. Ablowitz MJ, Fokas AS. *Complex variables: introduction and applications*. Cambridge University Press. 2003;2nd ed:1-647.
10. Needham T. *Visual complex analysis*. Oxford University Press. 1997;1:1-592.
11. Crowdy DG. Analytical formulae for the Kirchhoff-Routh path function in multiply connected domains. *Proceedings of the Royal Society A*. 2010;466(2120):2381-2403.
12. Murakami Y. *Stress intensity factors handbook*. Pergamon Press. 1987;1:1-1456.
13. Darbyshire AL, Bjarnason IT, White RS, Flovenz OG. Crustal structure above the Iceland mantle plume imaged by the ICEMELT refraction profile. *Geophysical Journal International*. 1998;135(3):1131-1149.
14. Fasshauer GE. *Meshfree approximation methods with MATLAB*. World Scientific. 2007;1:1-500.
15. Driscoll TA, Trefethen LN. *Schwarz-Christoffel mapping*. Cambridge University Press. 2002;1:1-132.
16. Carslaw HS, Jaeger JC. *Conduction of heat in solids*. Oxford University Press. 1959;2nd ed:1-510.
17. Hochstadt H. *The functions of mathematical physics*. Dover Publications. 1986;1:1-322.