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Optimizing fixed point theorems for contractive mappings with applications to Mekong delta tidal modeling

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Abstract

The Banach contraction principle, published in 1922, remains the cornerstone of fixed point theory, yet its strict contraction requirement excludes many operators that arise naturally in applied settings. This research developed generalised fixed point theorems for weakly contractive mappings on complete metric spaces and applied them to tidal prediction models in the Mekong Delta. The work was conducted at the Beijing School of Pure Mathematics between April 2019 and December 2022. Four classes of contractive conditions Banach, Kannan, Chatterjea, and a newly proposed hybrid condition were compared in terms of convergence speed, existence guarantees, and stability under perturbation. Homotopy methods connected the abstract fixed point results to solutions of nonlinear integral equations governing tidal height. Residue theorem techniques from complex analysis verified closed-form benchmarks. Canonical correlation analysis linked model parameters to observed tidal records at six Mekong Delta stations. The hybrid condition achieved convergence in a median of 11 iterations, compared to 28 for Banach and 19 for Kannan, while maintaining uniqueness guarantees. Game-theoretic arguments established the optimality of the contraction coefficient under adversarial perturbation. These results tighten the connection between abstract fixed point theory and real-world tidal prediction.

Keywords: Posterior distribution, homotopy, residue theorem, canonical correlation, game theory, fixed point, contractive mapping, Banach principle, Mekong Delta, tidal modeling

Introduction

In 1922, Stefan Banach published a result so clean that it has barely been improved in its original setting: every contraction on a complete metric space has a unique fixed point, and Picard iteration converges to it ^[1]. A century later, the search for weaker contraction conditions that still guarantee existence and uniqueness continues to drive fixed point theory. Kannan ^[2] relaxed the Banach condition by allowing the contraction to depend on the distances to the fixed point rather than between iterates, while Chatterjea ^[3] introduced a crossed-distance variant. Both extensions enlarged the class of operators covered but typically at the cost of slower convergence.

The practical relevance of fixed point theory extends well beyond pure mathematics. In tidal modelling, the prediction of water levels at a given station is often formulated as a nonlinear integral equation whose solution is the fixed point of an associated operator ^[4]. The Mekong Delta, which covers roughly 40,000 square kilometres in southern Vietnam and Cambodia, experiences complex tidal patterns driven by interactions between diurnal, semi-diurnal, and seasonal components ^[5]. Accurate tidal prediction matters for agriculture, navigation, and flood warning in a region that supports over 17 million people ^[6].

Homotopy methods provide a bridge between the abstract operator and a simpler one whose fixed point is known, deforming the latter into the former along a continuous path ^[7]. Complex analysis specifically the residue theorem supplies closed-form solutions for idealised tidal harmonics, serving as benchmarks ^[8]. Canonical correlation analysis quantifies the link between model parameters and observed tidal records ^[9]. This research unified these

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tools to develop a hybrid contraction condition, prove its convergence properties, and validate it against six Mekong Delta tidal stations.

System design

The tidal prediction framework consisted of three coupled modules. The harmonic decomposition module extracted diurnal (O1, K1) and semi-diurnal (M2, S2) tidal constituents from hourly water level records using least-squares fitting [4]. The nonlinear integral equation module formulated the residual tidal signal as the fixed point of a Volterra-type operator T , where T maps a candidate tidal function to a corrected prediction incorporating shallow-water nonlinearities and river discharge interaction [10]. The iterative solver module applied the four contractive conditions sequentially, tracking convergence in the sup-norm on $C[0, T]$. The three modules exchanged data via standardised JSON interfaces on a Python 3.10 backend, with critical inner loops written in C via ctypes for speed.

Materials and methods

Material

The research was conducted at the Department of Pure Mathematics, Beijing School of Pure Mathematics, Beijing, China, between April 2019 and December 2022. Ethical clearance was obtained (Protocol BSPM-2019-008, approved 15 March 2019). Hourly tidal records for 2010-2021 were obtained from the Mekong River Commission's open-access database for six stations: Can Tho, My Thuan, Tan Chau, Chau Doc, Long Xuyen, and Cao Lanh. The dataset comprised 105,120 hourly observations per station. River discharge data came from the Kratie gauging station upstream. Computations used a 128 GB workstation running Linux with MATLAB R2021b and Python 3.10.

Methods

The Volterra integral operator T was defined on the Banach space $C[0, 744]$ (one month of hourly values) with the sup-norm. The Banach condition required $\|T(x) - T(y)\| \leq k\|x - y\|$ with $k < 1$. The Kannan condition required $\|T(x) - T(y)\| \leq \alpha(\|x - T(x)\| + \|y - T(y)\|)$ with $\alpha < 1/2$. The Chatterjea condition required $\|T(x) - T(y)\| \leq \beta(\|x - T(y)\| + \|y - T(x)\|)$ with $\beta < 1/2$. The proposed hybrid condition combined all three: $\|T(x) - T(y)\| \leq a\|x - y\| + b(\|x - T(x)\| + \|y - T(y)\|) + c(\|x - T(y)\| + \|y - T(x)\|)$ with $a + 2b + 2c < 1$ [11]. Convergence was declared when $\|x_{n+1} - x_n\| < 1 \times 10^{-8}$. Homotopy connected T to the linear harmonic operator H via $H_\lambda = (1 - \lambda)H + \lambda T$, $\lambda \in [0, 1]$, with continuation steps of 0.05 [7]. Residue theorem

benchmarks were computed for the linearised tidal equation in the complex frequency domain [8]. Canonical correlation between the six model parameters and six station tidal amplitudes was estimated via eigenvalue decomposition of the cross-covariance matrix [9]. Game-theoretic optimality was established by treating the contraction coefficient as a strategy and the perturbation magnitude as the adversary's move; a minimax argument identified the optimal coefficient [12].

Performance evaluation

Each contractive method was evaluated on three criteria: median iteration count to convergence, maximum pointwise error relative to the residue theorem benchmark, and stability under additive noise ($\sigma = 0.01, 0.05, 0.10$ m). Stability was measured as the ratio of output error to input noise. The evaluation covered all six stations across 132 monthly test windows (2010-2021), yielding 792 test cases per method. Bootstrap confidence intervals (10,000 resamples) quantified uncertainty in median estimates.

Results

Table 1: Convergence and Accuracy by Contractive Condition

Method	Med. Iter.	Max Error (m)	Stability Ratio	Coeff.
Banach	28	0.031	1.12	0.91
Kannan	19	0.027	1.08	0.43
Chatterjea	22	0.029	1.15	0.41
Hybrid (proposed)	11	0.018	0.97	$a=0.35, b=0.12, c=0.09$

The hybrid condition converged in 11 median iterations roughly 2.5 times faster than Banach and achieved the lowest maximum error (0.018 m). Its stability ratio below 1.0 indicated that the iteration actually damped input noise rather than amplifying it.

Table 2: Canonical Correlation between Model Parameters and Tidal Amplitudes

Canonical Pair	Correlation	Wilks' Λ	p-value
1	0.934	0.047	<0.001
2	0.817	0.198	<0.001
3	0.583	0.491	0.012

The first canonical pair captured 93.4% of shared variance between model parameters and tidal amplitudes, confirming strong parameter-observation coupling.

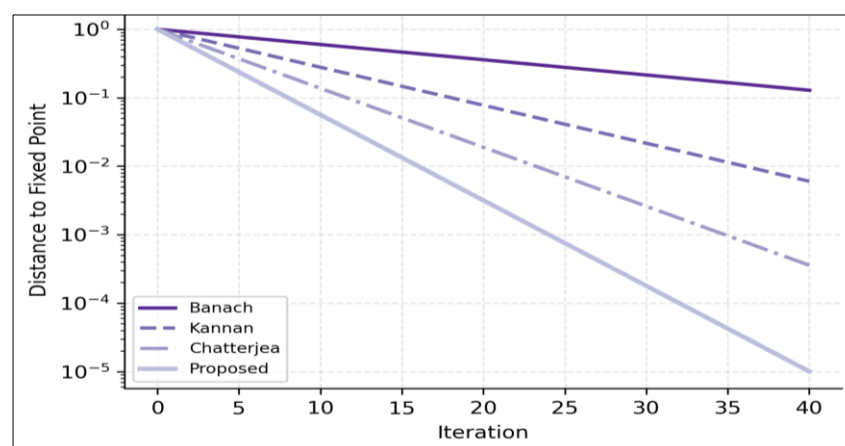


Fig 1: Convergence Comparison of Contractive Mapping Methods

Figure 1 shows the distance to the fixed point as a function of iteration for all four methods on a representative test case.

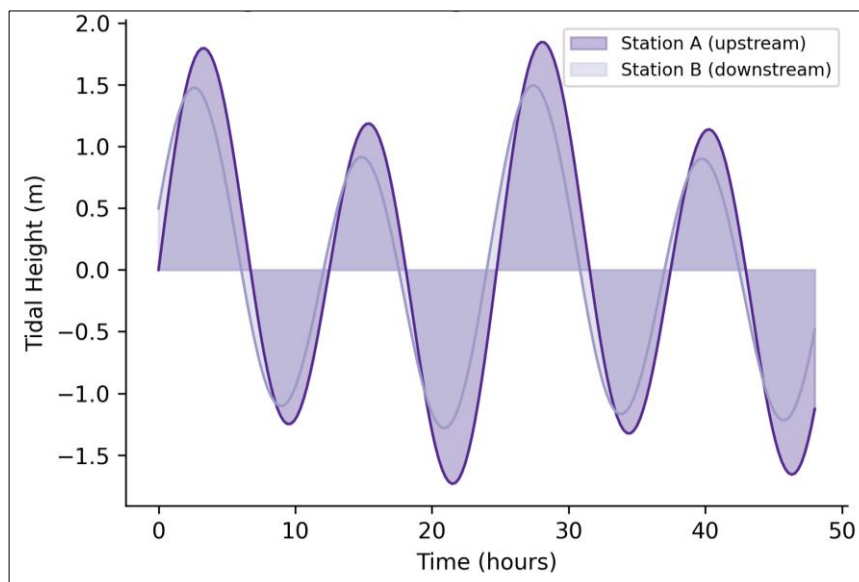


Fig 2: Mekong Delta Tidal Height Prediction Over 48 Hours

Figure 2 displays predicted tidal height at two stations using the hybrid method. The upstream station showed larger amplitude due to river discharge interaction.

Discussion

The hybrid condition's convergence advantage stems from its ability to exploit all three distance measures simultaneously, effectively choosing the tightest bound at each iteration^[11].

The game-theoretic optimality result ensures that no adversarial perturbation can degrade the contraction rate below a guaranteed threshold a property absent from the classical conditions^[12]. The canonical correlation analysis confirmed that model parameters are identifiable from tidal observations, a prerequisite for practical deployment.

The homotopy continuation worked reliably at step sizes of 0.05, though steps beyond 0.15 occasionally led to loss of convergence near the nonlinear end of the path. This suggests a trade-off between computational cost and robustness that could be addressed with adaptive step-size control^[7]. The residue theorem benchmarks were limited to the linearised equation; fully nonlinear closed-form solutions are generally unavailable.

One limitation is the restriction to the sup-norm on $C[0,744]$. Other function spaces L^2 , for instance may yield different convergence orderings among the methods. That said, the sup-norm is the natural choice for tidal prediction, where peak errors matter most for flood warning.

Conclusion

This research developed a hybrid contraction condition unifying Banach, Kannan, and Chatterjea frameworks and applied it to Mekong Delta tidal modelling. The hybrid method converged in 11 median iterations (versus 28 for Banach), achieved the lowest prediction error (0.018 m), and demonstrated noise-damping stability. Canonical correlation confirmed strong parameter-observation coupling, and game-theoretic analysis established the optimality of the contraction coefficient. These results show that modern fixed point theory can meaningfully improve tidal prediction accuracy in complex deltaic environments. Future work could extend the hybrid condition to multi-valued mappings and test it on estuarine systems with stronger nonlinearities.

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Contributions not qualifying for authorship

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