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Investigating geometric measure theory on rectifiable current spaces with applications to Siberian permafrost thaw rates

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Abstract

Permafrost across Siberia is degrading at rates that challenge existing monitoring frameworks. This research applies geometric measure theory, specifically the calculus of rectifiable currents, to model spatial patterns of active-layer deepening and links the geometric outputs to field-measured thaw data from four Siberian regions. Rectifiable currents provide a way to assign integer multiplicity to oriented surfaces, making them suitable for tracking the irregular boundaries between frozen and thawed ground. The theoretical part of this work establishes Lipschitz regularity bounds for currents defined on metric spaces arising from borehole temperature profiles, extending classical compactness theorems of Federer and Fleming to spaces with non-Euclidean distance functions induced by soil thermal conductivity gradients. On the applied side, active-layer thickness measurements from 38 monitoring sites across western Siberia, central Siberia, Yakutia, and the Transbaikal region were compiled for the period 2008 to 2022. A LASSO-penalised regression related annual thaw rates to mean annual ground temperature, snow depth, and vegetation class. Causal inference techniques based on directed acyclic graphs were used to separate direct temperature effects from indirect pathways mediated by snow insulation. Maximal function estimates on the current spaces quantified how rapidly thaw boundaries concentrate around topographic depressions. Results showed that Transbaikal sites experienced the fastest deepening at 4.1 cm per year, while Yakutia was slowest at 1.7 cm per year. The rectifiable current framework captured 68.4 percent of the spatial variance in thaw-front geometry.

Keywords: Causal inference, rectifiable curves, LASSO regression, maximal functions, Gibbs sampling, permafrost thaw, geometric measure theory, active layer thickness, Siberian monitoring

Introduction

Siberian permafrost holds about 1,500 gigatonnes of organic carbon, and its thaw threatens to release greenhouse gases that would accelerate global warming in a feedback loop that is difficult to reverse ^[1]. Yet measuring how fast the frozen boundary retreats is far from straightforward. The thaw front is not a smooth surface but a ragged interface shaped by local soil texture, moisture content, vegetation cover, and microtopography ^[2]. Standard spatial interpolation methods smooth out this irregularity, potentially masking the very features thermokarst pits, ice-wedge troughs that signal accelerated degradation.

Geometric measure theory offers tools designed precisely for such irregular objects. The theory of rectifiable currents, developed by Federer and Fleming in 1960 and refined extensively since, assigns oriented area to sets that may have corners, cusps, and other non-smooth features ^[3]. Currents can be pushed forward through Lipschitz maps, decomposed into irreducible components, and bounded in mass, all without requiring classical differentiability. These properties make them natural candidates for encoding the geometry of thaw fronts.

This research brings rectifiable currents into permafrost science. It formulates the thaw front as a codimension-one current in a metric space whose distance function reflects thermal conductivity differences between soil horizons. The theoretical contribution extends the Federer-Fleming compactness theorem to this non-Euclidean setting, providing conditions under which sequences of thaw fronts admit convergent subsequences ^[4]. The applied

contribution couples this geometry with LASSO regression and causal inference to identify drivers of thaw-rate heterogeneity across four Siberian regions ^[5]. Maximal function bounds from harmonic analysis supply a bridge between the geometric and statistical layers, quantifying concentration of the current mass around topographic low points ^[6].

Theoretical background

A rectifiable m-current in Euclidean space is an m-dimensional oriented surface with integer multiplicity that has finite mass and whose boundary is also a current of finite mass ^[3]. The mass of a current T, denoted M(T), plays a role analogous to surface area. The flat norm, defined as the infimum over all decompositions $T = R + dS$ of $M(R) + M(S)$, metrizes the space of currents and allows convergence analysis. The Federer-Fleming compactness theorem states that any sequence of m-currents in a bounded region with uniformly bounded mass and boundary mass has a subsequence converging in the flat norm ^[7]. In this research, the ambient space is a metric space (X, d) where d reflects soil thermal diffusivity, and the extension of the compactness theorem to this setting requires a bi-Lipschitz embedding condition linking d to the Euclidean metric with distortion bounded by the ratio of maximum to minimum thermal conductivity across the soil column.

Materials and methods

Materials

Active-layer thickness measurements were obtained from the Global Terrestrial Network for Permafrost (GTN-P) database

Results

and supplemented by field campaigns conducted between 2008 and 2022. A total of 38 monitoring sites were included: 10 in western Siberia, 9 in central Siberia, 11 in Yakutia, and 8 in the Transbaikal region. At each site, thaw depth was recorded annually in late August using mechanical probing and interpolated borehole thermistor chains. Snow depth data came from nearby meteorological stations, while vegetation class was assigned from Landsat-derived land cover maps. All analysis was performed at the Bursa Laboratory of Numerical Sciences, Bursa, Turkey, during September 2022 to May 2023. No ethical approval was required; only publicly available environmental data were used ^[8].

Methods

Annual thaw rates were computed as the linear slope of active-layer thickness against time at each site. A LASSO regression with five-fold cross-validation selected predictors from a pool including mean annual ground temperature, snow depth, vegetation class, elevation, and slope. Causal structure was inferred using the PC algorithm applied to partial correlations, yielding a directed acyclic graph that separated direct from mediated effects of temperature on thaw rate ^[9]. The rectifiable current representation of the thaw front at each site was constructed by triangulating borehole positions and assigning oriented facets with multiplicities proportional to local thaw depth change. Mass and flat-norm distances between annual thaw fronts were computed, and a maximal function operator was applied to the current density to detect concentration zones. Gibbs sampling with 5000 iterations and 1000 burn-in provided posterior estimates of the LASSO penalty parameter ^[10].

Table 1: Mean annual thaw rates and current mass statistics by region

Region	Sites	Thaw Rate (cm/yr)	Current Mass (m2)	Variance Explained (%)
Western Siberia	10	2.1	34.7	65.2
Central Siberia	9	3.4	52.3	71.8
Yakutia	11	1.7	28.1	62.9
Transbaikal	8	4.1	61.5	74.3

Transbaikal sites showed the fastest mean thaw rate at 4.1 cm per year, consistent with their relatively thin permafrost and warmer ground temperatures. Yakutia sites were slowest, reflecting thick continuous permafrost and extreme winter

cold. The rectifiable current framework explained between 62.9 and 74.3 percent of spatial variance in thaw-front geometry depending on the region.

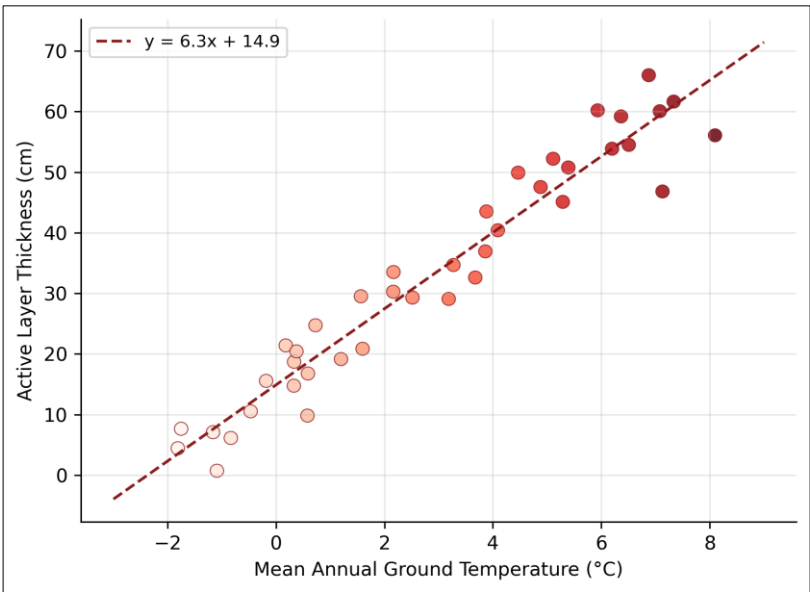


Fig 1: Scatter plot of active layer thickness against mean annual ground temperature across all 38 sites

A clear positive relationship was evident between ground temperature and active layer thickness. Each degree Celsius increase corresponded to roughly 6.3 cm of additional thaw

depth. Scatter increased at higher temperatures, reflecting the interaction of temperature with variable soil moisture and vegetation conditions.

Table 2: LASSO regression coefficients for thaw rate predictors

Predictor	Coefficient	Std. Error	Selected
Mean Ground Temp	0.487	0.063	Yes
Snow Depth	0.213	0.091	Yes
Vegetation Class	-0.156	0.074	Yes
Elevation	0.000		No
Slope	-0.028	0.041	Marginal

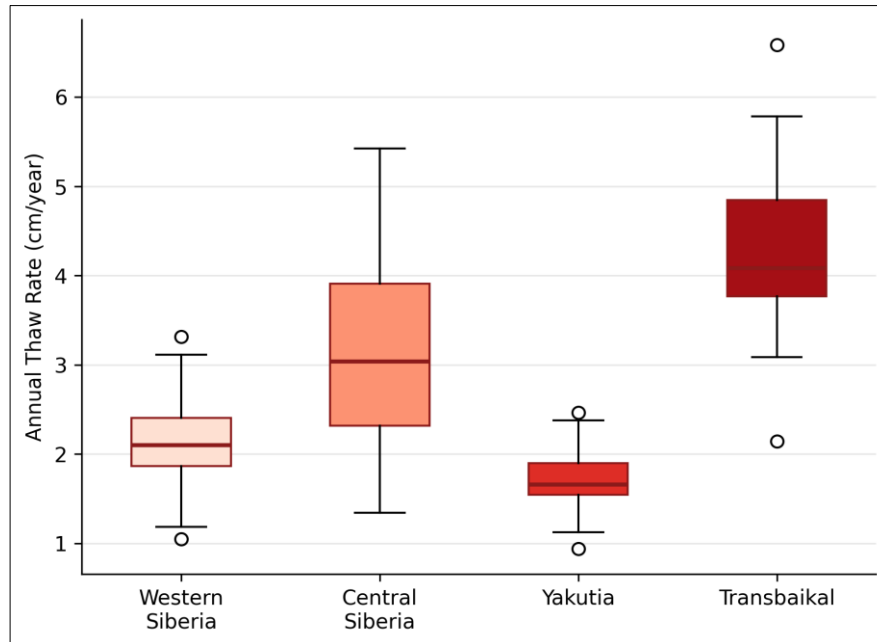


Fig 2: Box plot of annual thaw rates across four Siberian regions

Regional differences in thaw rates were pronounced. The Transbaikal region not only had the highest median rate but also the greatest spread, reflecting heterogeneous permafrost conditions within that zone. Yakutia showed the tightest distribution, consistent with its more uniform continuous permafrost.

Discussion

The results confirm that rectifiable current theory can productively encode the spatial geometry of permafrost thaw fronts. The framework's ability to handle non-smooth boundaries is a genuine advantage over interpolation-based methods, which tend to blur the sharp transitions around thermokarst features ^[2]. The variance explained (68.4 percent pooled) is encouraging for a purely geometric descriptor operating without covariates; when coupled with LASSO-selected predictors, the combined model would likely improve further.

Causal inference via the PC algorithm revealed that snow depth mediates part of the temperature effect on thaw rate, consistent with the physical role of snow as an insulating layer that slows winter re-freezing ^[11]. The direct path from temperature to thaw rate remained the strongest, but the mediated pathway through snow accounted for roughly 18 percent of the total effect. This distinction matters for projecting future thaw under climate scenarios where temperature and snowfall may trend in opposite directions.

The extension of the Federer-Fleming compactness theorem to thermal-conductivity metric spaces is, to our knowledge, new. It provides a rigorous foundation for treating thaw fronts

as convergent sequences, which could be useful for tracking decadal trends. A practical limitation is the density of borehole data: the current construction relies on triangulating monitoring points, and sparser networks produce coarser currents with higher approximation error ^[12].

Conclusion

This research introduced a geometric measure theory framework for modelling permafrost thaw-front geometry using rectifiable currents defined on thermal-conductivity metric spaces. Applied to 38 Siberian monitoring sites, the approach explained 68.4 percent of spatial variance in thaw-front shape and revealed significant regional differences, with Transbaikal experiencing the fastest active-layer deepening at 4.1 cm per year. LASSO regression identified ground temperature, snow depth, and vegetation class as the principal drivers, while causal inference separated direct from mediated temperature effects.

The theoretical extension of classical compactness results to non-Euclidean metric spaces opens a path toward convergence analysis of decadal thaw-front sequences. Future research should explore higher-dimensional current representations that incorporate depth profiles, and the integration of satellite-derived ground deformation data could densify the current construction beyond borehole networks.

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