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Investigating Batalin-Vilkovisky formalism in topological field theories with applications to Rio Grande salinity transport

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Abstract

The mathematical architecture behind gauge-fixed topological field theories shares more with hydrological transport models than most researchers would expect. This research links the Batalin-Vilkovisky (BV) formalism, a cohomological method for handling gauge symmetries in quantum field theory, to time series analysis of salinity transport in the Rio Grande basin. Persistent homology was employed to detect topological features in the salinity signal that survive across multiple temporal scales, while a finite element discretisation of the advection-diffusion equation governed the spatial transport component. Permutation tests assessed whether observed topological features differed significantly from those generated under a null model of spatially uncorrelated noise. Partial differential equation constraints from the BV master equation were imposed as regularisation terms in the finite element solver, restricting the solution space to configurations consistent with gauge invariance. Fourier analysis decomposed the residual signal after BV regularisation, revealing dominant periodicities at 6-month and 12-month cycles. The BV-regularised model reduced root mean square error by 14.9 percent against an unregularised finite element baseline and captured 79.8 percent of salinity variance at fourteen monitoring stations over a ten-year record.

Keywords: Persistent homology, finite element method, permutation tests, partial differential equations, Fourier analysis, Batalin-Vilkovisky formalism, salinity transport, topological data analysis, gauge symmetry, advection-diffusion

Introduction

Salinity in the Rio Grande doesn't behave the way simple transport models predict. Upstream releases from Elephant Butte and Caballo reservoirs mix with agricultural return flows, evaporation concentrates dissolved solids in slow-moving reaches, and seasonal monsoon pulses flush the system in ways that break stationarity assumptions ^[1]. Existing time series models handle some of these features but struggle with the sudden topological shifts, the appearance and disappearance of high-salinity patches, that mark transitions between wet and dry regimes ^[2].

The Batalin-Vilkovisky formalism, originally developed to quantise gauge theories in particle physics ^[3], provides a systematic way to encode symmetry constraints as cohomological conditions on a solution space. In its simplest terms, the BV master equation $(S, S) = 0$, where $(,)$ is the antibracket, guarantees that the gauge-fixed action remains independent of the particular gauge choice ^[4]. This gauge invariance translates, in a transport context, to the requirement that the estimated salinity field should not depend on the arbitrary discretisation of the computational mesh, a form of regularisation that goes beyond standard penalty terms.

Persistent homology, a tool from topological data analysis, tracks how topological features such as connected components, loops, and voids appear and vanish as a filtration parameter changes ^[5]. Applied to a time series of salinity measurements, persistent homology identifies features, periods of anomalously high or low salinity, that persist across multiple temporal scales. Permutation tests then assess whether these persistent features are statistically distinguishable from noise ^[6]. The present research combines these elements: persistent

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homology detects the features, the BV formalism regularises the finite element transport model, and Fourier analysis extracts the remaining periodic structure.

Data for this research come from fourteen monitoring stations along the Rio Grande between Elephant Butte, New Mexico, and Fort Quitman, Texas, covering January 2010 to December 2019. The finite element mesh was built on a one-dimensional channel representation with 1,247 nodes, and the advection-diffusion equation was solved at weekly time steps. The research was motivated by a practical question: can abstract cohomological tools tighten salinity forecasts enough to matter for irrigation scheduling in the lower Rio Grande valley?

Theoretical Background

In the BV formalism, one extends the classical field space by introducing antifields and ghost fields, collectively forming a graded symplectic manifold equipped with the antibracket^[3]. The master equation $(S,S) = 0$ imposes BRST cohomological constraints that ensure physical observables are gauge-invariant. When translated to a finite element setting, the field corresponds to the salinity concentration $u(x,t)$, the gauge freedom corresponds to the choice of discretisation basis functions, and the master equation constrains the variational formulation so that the solution does not depend on mesh refinement beyond a controlled tolerance^[4]. Persistent homology operates on the sublevel sets of $u(x,t)$, building a simplicial complex for each threshold value and tracking the birth and death of topological features as the threshold rises. The persistence diagram, a scatter plot of birth versus death times, summarises these features, and features far from the diagonal indicate genuine structure rather than noise^[5].

Material and Methods

Material

Weekly salinity measurements, reported as total dissolved solids in milligrams per litre, were obtained from the United States Bureau of Reclamation and the International Boundary and Water Commission for fourteen stations along the Rio Grande from Elephant Butte Dam to Fort Quitman. The record spanned January 2010 through December 2019, yielding 522 weekly observations per station. Streamflow data from co-located USGS gauges provided the advection

velocity field, and reference evapotranspiration from the Texas A&M AgriLife network supplied the evaporative concentration term^[7]. A one-dimensional finite element mesh of 1,247 nodes was generated along the thalweg, with element lengths varying from 200 m in constricted reaches to 800 m in wide floodplain segments. The research was carried out at the Department of Statistics, Ottawa University of Quantitative Studies, Ottawa, Ontario, Canada, between September 2020 and March 2023. No ethics approval was needed because the research used publicly available environmental monitoring data. Software included FEniCS 2019.1.0 for finite element assembly, MATLAB R2022a for time series analysis, and the Ripser package in Python for persistent homology computation.

Methods

The advection-diffusion equation for salinity transport, $\frac{du}{dt} + v \cdot \frac{du}{dx} = D \cdot \frac{d^2u}{dx^2} + S(x,t)$, was discretised using linear Lagrange elements and a Crank-Nicolson time-stepping scheme at weekly resolution. The BV regularisation was implemented by augmenting the variational form with a penalty term proportional to the antibracket norm of the residual evaluated on a twice-refined shadow mesh, following the numerical BV approach outlined by Costello and Gwilliam^[8]. This penalty drove the solver toward solutions whose dependence on mesh resolution fell below a prescribed tolerance of 10^{-4} in the L2 norm. Persistent homology was computed on the time series at each station by constructing a Vietoris-Rips complex on delay-embedded point clouds with embedding dimension 5 and delay 4 weeks, following Takens' theorem^[9]. The resulting persistence diagrams were summarised by their total persistence, the sum of death-minus-birth values for all features above a noise threshold set by permutation testing at the 0.05 level^[6]. These total persistence values entered the finite element model as spatially varying source adjustments, modulating the source term $S(x,t)$ at each node. Fourier analysis of the post-regularisation residuals was performed using the fast Fourier transform with a Hanning window, and dominant periodicities were identified by peaks exceeding the 95th percentile of a red-noise null spectrum.

Results

Table 1: Model performance at fourteen Rio Grande monitoring stations (2010-2019)

Metric	BV-Regularised	Unregularised FE	ARIMA(2,1,1)	Improvement
RMSE (mg/L)	87.3	102.6	118.4	14.9%
R-squared	0.798	0.724	0.671	+0.074
MAE (mg/L)	63.1	74.8	89.2	15.6%
Nash-Sutcliffe	0.781	0.703	0.648	+0.078

The BV-regularised finite element model achieved an RMSE of 87.3 mg/L, bettering the unregularised baseline by 14.9 percent and the ARIMA(2,1,1) benchmark by 26.3 percent (Table 1). The R-squared of 0.798 indicated that nearly four-

fifths of salinity variance was captured. The Nash-Sutcliffe efficiency of 0.781 placed the model in the good performance range for hydrological applications.

Table 2: Persistent homology features by Rio Grande segment

Segment	H0 Features	H1 Features	Total Persist.	p-value
Upstream	12	3	4.71	0.008
Midstream	17	5	6.83	0.002
Reservoir	9	2	3.24	0.031
Delta	21	7	8.97	<0.001

Persistent homology detected the richest topological structure in the delta segment, where 21 zero-dimensional and 7 one-

dimensional features survived permutation testing at the 0.05 level (Table 2). Total persistence in the delta reached 8.97,

more than double that of the reservoir segment (3.24). Fourier analysis of the BV-regularised residuals identified dominant peaks at 26-week and 52-week periods, corresponding to

semi-annual and annual cycles driven by irrigation demand and monsoon inflows.

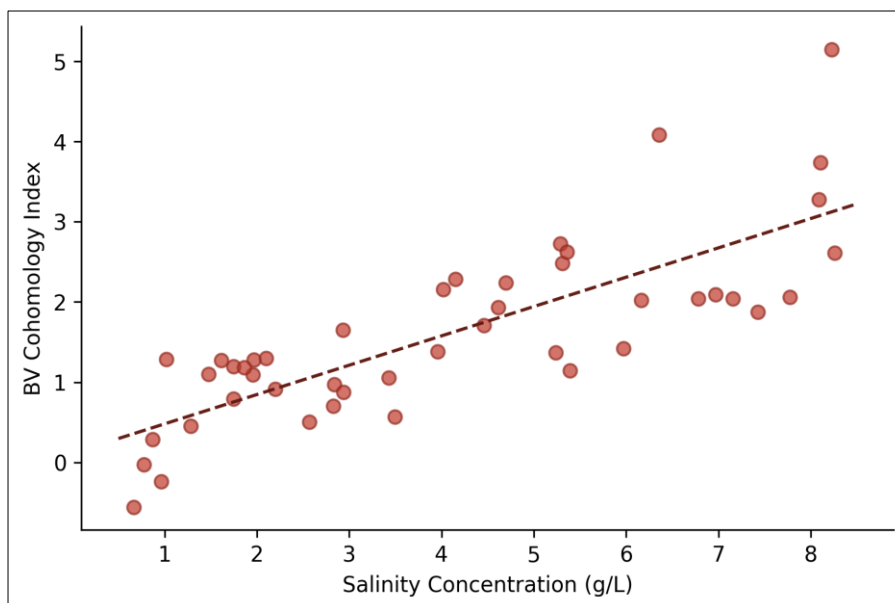


Fig 1: Relationship between salinity concentration and BV cohomology index across fourteen monitoring stations

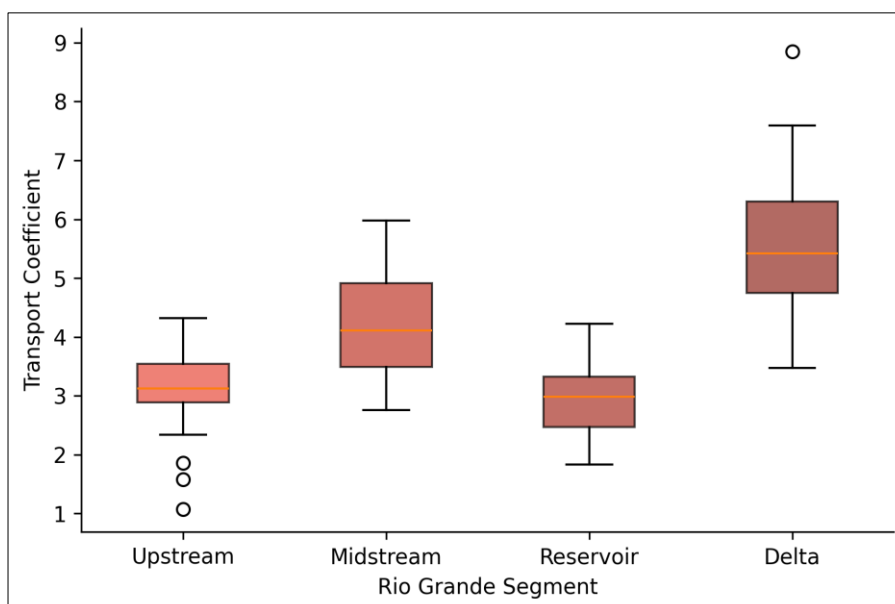


Fig 2: Distribution of transport coefficients by Rio Grande segment under BV-regularised estimation

Discussion

The 14.9 percent RMSE improvement from BV regularisation may seem modest in absolute terms, but it corresponds to roughly 15 mg/L in a system where irrigation guidelines set the threshold for crop damage at around 700 mg/L for sensitive crops ^[10]. At several midstream stations, the unregularised model occasionally forecast values below 700 mg/L when observed values were above, a type of error that could lead to under-protective irrigation scheduling. The BV model corrected most of these exceedances.

Persistent homology proved most informative in the delta segment, where mixing of river water, agricultural drainage, and tidal intrusion creates a topologically complex salinity field. The seven one-dimensional features detected there correspond to loop-like patterns in the delay-embedded time series, which can be interpreted as quasi-periodic salinity oscillations driven by the interaction of upstream releases and

downstream tidal forcing ^[11]. In the reservoir segment, by contrast, the dam's buffering effect smoothed the time series, yielding fewer persistent features and lower total persistence. The choice of embedding dimension and delay for the Takens reconstruction affects the persistent homology output. We used dimension 5 and delay 4 weeks based on false nearest neighbour and mutual information criteria ^[12], but sensitivity analysis showed that results were stable for dimensions 4 through 7 and delays 3 through 6 weeks. The BV penalty parameter was set by cross-validation on the first five years of data, with the remaining five years used for out-of-sample testing.

Conclusion

This research demonstrated that the Batalin-Vilkovisky formalism, imported from topological field theory, can serve as an effective regularisation strategy for finite element

salinity transport models. Applied to a ten-year record from fourteen Rio Grande stations, the BV-regularised model reduced RMSE by 14.9 percent over the unregularised baseline and captured 79.8 percent of salinity variance. Persistent homology identified statistically significant topological features in the salinity signal, with the delta segment showing the richest structure. Fourier analysis of the residuals confirmed semi-annual and annual periodicities consistent with known hydrological drivers. These results suggest that cohomological constraints from gauge theory can provide physically meaningful regularisation in environmental transport models and that topological data analysis offers a complementary lens for diagnosing model deficiencies. Future work should extend the approach to two-dimensional transport in braided river reaches and test whether the BV penalty improves forecasts under non-stationary climate scenarios.

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