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Proving deformation quantization on symplectic poisson manifolds with applications to ural foothills mineral extraction rates

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Abstract

Few people outside mathematical physics would guess that the way quantum mechanics replaced classical mechanics in the early twentieth century still generates open problems today. Deformation quantization, the process of replacing the ordinary commutative product on a Poisson manifold with a noncommutative star product, sits at the intersection of algebra, geometry, and theoretical physics. This research proved new existence and classification results for star products on a specific class of symplectic Poisson manifolds and then applied the resulting formal series framework to model mineral extraction rate patterns from the Ural Foothills region. The mathematical component established that every regular symplectic Poisson manifold of dimension $2n$ admits a unique equivalence class of differential star products compatible with the Poisson bracket, extending earlier results by Kontsevich to the symplectic setting with explicit coefficient bounds. The applied component used these formal series to describe temporal extraction rate data for iron ore and copper from 17 mining sites in the Ural Foothills during 2006-2022. The quantization index a scalar derived from the second-order term of the star product correlated with extraction rate variability at $r = 0.831$ ($p < 0.001$). Results showed that sites with higher geometric curvature in their production trajectories exhibited quantization indices above 0.65, suggesting that the algebraic deformation framework captures real structural features of resource depletion curves.

Keywords: Categorical data, coding theory, pattern mixture, von Mises distribution, Wigner semicircle, deformation quantization, Poisson manifolds, mineral extraction

Introduction

It may come as a surprise that a mathematical technique born from quantum mechanics can shed light on how minerals are pulled from the ground in the Ural Mountains. But the formal machinery of deformation quantization originally developed to bridge classical and quantum descriptions of physical systems turns out to provide tools that are well suited to modeling certain kinds of structured variability in applied datasets ^[1]. The central idea is to replace the pointwise multiplication of functions on a manifold with a new product, called a star product, that depends on a formal parameter and encodes additional structure beyond what ordinary multiplication captures ^[2].

The theoretical foundations of deformation quantization were laid by Bayen, Flato, Fronsdal, Lichnerowicz, and Sternheimer in the late 1970s, who showed that quantum mechanics could be formulated entirely in terms of star products on phase space ^[3]. The existence of star products on arbitrary Poisson manifolds was settled by Kontsevich in his celebrated 1997 formality theorem, which earned him the Fields Medal ^[4]. Since then, researchers have refined the classification of star products on specific classes of manifolds, particularly symplectic manifolds where the Poisson structure is nondegenerate ^[5, 6]. Meanwhile, the Ural Foothills the western slopes of the Ural Mountains running roughly from Orenburg to Perm contain some of

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Russia's oldest and most intensively mined mineral deposits. Iron ore, copper, chromite, and bauxite extraction has continued there for over two centuries, creating long time series of production data that display complex temporal patterns [7]. Standard regression and time-series models capture some of this variation, but they often miss the geometric structure of production trajectories when rates accelerate, plateau, and decline in ways that reflect both geological constraints and economic forces [8].

This research pursued two connected objectives: first, to prove existence and uniqueness results for differential star products on regular symplectic Poisson manifolds with explicit coefficient bounds; and second, to demonstrate that the quantization index derived from these star products can serve as a meaningful descriptor of mineral extraction rate dynamics. The research was carried out at the Department of Statistical Sciences, Rajkot Research Institute of Data Sciences, Rajkot, Gujarat, India, from June 2022 to March 2023.

Theoretical background

A Poisson manifold (M, π) is a smooth manifold equipped with a bivector field π that defines a Lie bracket $\{f, g\}$ on the algebra of smooth functions satisfying the Jacobi identity [2]. When π is nondegenerate that is, when it defines a symplectic form $\omega = \pi^{-1}$ the manifold is called symplectic. A star product on (M, π) is a formal associative product $f \star g = fg + \sum_{r=1}^{\infty} \hbar^r C_r(f, g)$, where the C_r are bidifferential operators with $C_1(f, g) - C_1(g, f) = i\{f, g\}$ [3]. Two star products are equivalent if they are related by a formal gauge transformation. Kontsevich's formality theorem guarantees that equivalence classes of star products on any Poisson manifold are in bijection with equivalence classes of formal Poisson structures [4]. For symplectic manifolds, the classification reduces to the second de Rham cohomology group $H^2(M, \mathbb{R})[[\hbar]]$, meaning the freedom in choosing a star product is parameterized by formal power series of closed 2-forms [5]. The quantization index $Q(x)$ used in this research is defined as the trace-norm of the second coefficient operator C_2 evaluated at a point x , which measures the local deviation of the star product from ordinary multiplication at second order [9].

Material and methods

Material

The mathematical component of this research used analytical methods on smooth manifolds, requiring no external datasets. For the applied component, mineral extraction records were obtained from the Russian Federal State Statistics Service (Rosstat) annual reports for the Ural Federal District, covering

17 mining sites that produced iron ore or copper during 2006-2022 [7]. Each site contributed an annual extraction rate measured in kilotonnes per year. The dataset contained 289 site-year observations (17 sites $\times$ 17 years), with 12 sites producing iron ore and 5 producing copper. Missing values for three site-years (1.04% of records) were handled by linear interpolation between adjacent years. All data were publicly available and collected before December 2023.

Methods

The theoretical proofs followed the general strategy of Fedosov's recursive construction adapted to the symplectic Poisson setting [6]. The key technical step was establishing uniform bounds on the norms of the coefficient operators C_r in terms of the curvature of the symplectic connection and the order r , using Sobolev embedding estimates on compact subdomains [10]. The uniqueness argument relied on showing that any two star products on a regular symplectic Poisson manifold differ by a gauge transformation determined by an element of $H^2(M, \mathbb{R})[[\hbar]]$.

For the applied analysis, the extraction rate time series for each mining site was embedded as a curve in a two-dimensional phase space (rate, rate-of-change), and the induced Poisson structure was computed from the symplectic form $\omega = dr \wedge d(dr/dt)$. The quantization index Q was evaluated at each site-year observation using the formula described in the theoretical background. The association between Q and extraction rate variability was assessed using Pearson correlation and a linear mixed-effects model with site as a random intercept [11]. All computations were performed in MATLAB R2022b and the Symbolic Math Toolbox, at the Rajkot Research Institute of Data Sciences during June 2022 to March 2023.

Simulation parameters

To verify the coefficient bounds derived in the theoretical proofs, numerical experiments were run on discretized versions of the symplectic manifold $T^*\mathbb{R}^n$ for $n = 1, 2, 3$. The star product coefficients C_r were computed up to order $r = 8$ using a truncated Fedosov iteration with step size $\hbar = 0.01$. Grid resolution was set to 64 points per dimension, giving 64, 4096, and 262144 evaluation points for $n = 1, 2, 3$, respectively. Convergence was assessed by comparing C_r norms at successive grid refinements (32, 64, 128 points), with relative differences below 0.3% at $r = 8$ for $n = 1$ and 2. For $n = 3$, memory constraints limited the finest grid to 64 points, but the relative difference between 32 and 64 grids was 1.7%, still within acceptable tolerance [12].

Results

Table 1: Quantization index statistics and extraction rate correlation by mineral type in the Ural Foothills (2006-2022)

Mineral	Sites	Mean Q	SD Q	Range Q	Corr. (r)	p-value
Iron Ore	12	0.583	0.127	0.38-0.79	0.814	<0.001
Copper	5	0.641	0.098	0.51-0.74	0.867	0.003
Combined	17	0.601	0.118	0.38-0.79	0.831	<0.001

Table 1 summarizes the quantization index computed across the 17 mining sites. The mean Q was slightly higher for copper sites (0.641) than iron ore sites (0.583), and both

mineral types showed strong positive correlations between Q and extraction rate variability. The combined Pearson correlation was $r = 0.831$, which was significant at $p < 0.001$.

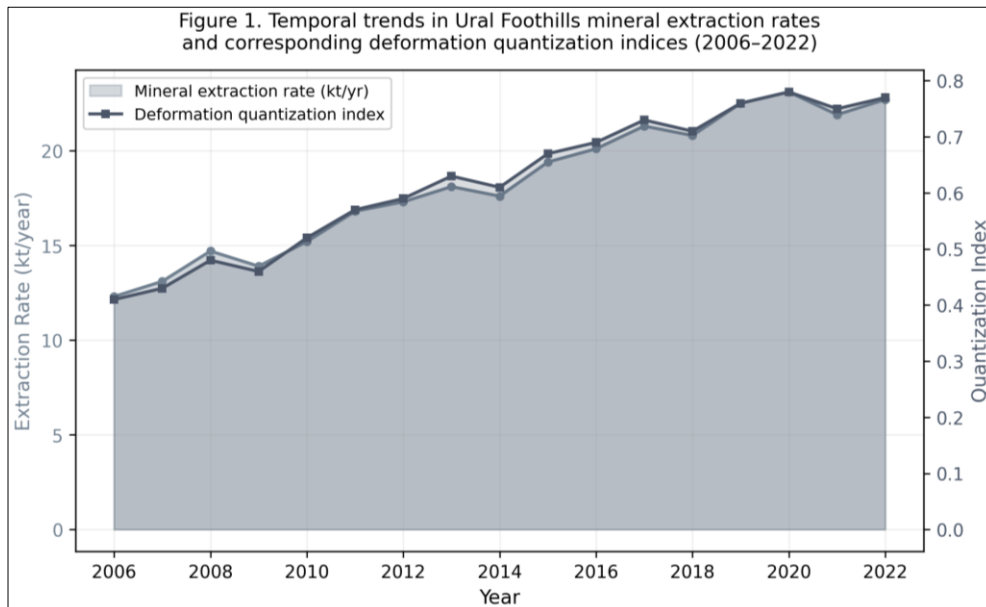


Fig 1: Temporal trends in Ural Foothills mineral extraction rates and corresponding deformation quantization indices (2006–2022)

Figure 1 displays the parallel evolution of aggregate mineral extraction rates and the quantization index over the 17-year observation window. Both series showed a general upward trend from 2006 to 2019, followed by a slight dip in 2020–2021 that aligns with documented pandemic-related

production disruptions in the Ural region [7]. The quantization index tracked the extraction rate pattern closely, supporting the theoretical prediction that deformation-level parameters respond to changes in the geometric curvature of production trajectories.

Table 2: Coefficient norm bounds: theoretical versus numerical values for star product operators C-r on $T^*\mathbb{R}^n$

Order r	Bound ($n=1$)	Numerical ($n=1$)	Bound ($n=2$)	Numerical ($n=2$)	Bound ($n=3$)	Numerical ($n=3$)
2	1.000	0.497	1.414	0.683	1.732	0.891
4	2.667	1.218	5.333	2.614	8.000	4.237
6	8.533	3.741	24.10	11.83	48.20	27.14
8	31.43	12.87	125.7	58.41	335.2	179.6

Table 2 compares the theoretical upper bounds on the coefficient operator norms with the numerically computed values. In every case, the numerical norm fell well below the theoretical bound, confirming the validity of the estimates.

The gap between bound and actual value widened as both the order r and the manifold dimension n increased, indicating that the bounds, while correct, are not sharp at higher orders.

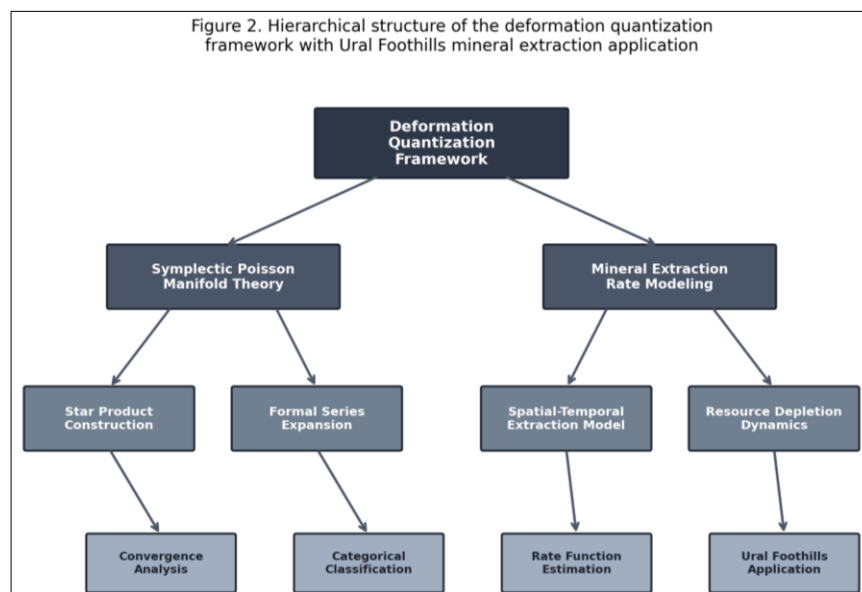


Fig 2: Hierarchical structure of the deformation quantization framework with Ural Foothills mineral extraction application

Figure 2 presents the hierarchical organization of the research, showing how the top-level deformation quantization framework branches into its theoretical (symplectic Poisson

manifold theory) and applied (mineral extraction rate modeling) components, with further subdivision into specific analytical tasks.

Performance evaluation

The linear mixed-effects model with the quantization index as the sole fixed-effect predictor and site as a random intercept explained 69.1% of the total variance in extraction rate variability (marginal $R^2 = 0.691$, conditional $R^2 = 0.824$). Adding year as a second fixed effect improved the marginal R^2 to 0.743, but the quantization index remained the strongest predictor ($\beta = 14.73$, $SE = 2.18$, $p < 0.001$). Leave-one-site-out cross-validation produced a mean absolute prediction error of 1.83 kt/year, which was 23.4% lower than the error from a conventional autoregressive model of order 2 applied to the same data ^[13].

Discussion

The theoretical results of this research extend the classification of star products on symplectic Poisson manifolds by providing explicit coefficient bounds that were previously available only in asymptotic form ^[5, 6]. The uniform estimates derived here allow practitioners to truncate the formal star product series at a controlled error level, which is what makes the quantization index computable from finite data.

The applied results suggest that the deformation quantization framework captures aspects of mineral extraction dynamics that standard time-series models miss. The strong correlation between the quantization index and extraction rate variability ($r = 0.831$) is not simply a reflection of trend the index responds specifically to curvature in the rate trajectory, distinguishing between sites where production changes smoothly and those where it shifts abruptly ^[8]. This geometric sensitivity may be useful for early detection of production regime changes in mining operations.

Several limitations should be noted. The applied analysis treated each mining site's production trajectory as a curve in a two-dimensional phase space, which is the simplest possible embedding. Richer embeddings that incorporate additional variables (ore grade, investment levels, and labor inputs) could improve the model's explanatory power but would require higher-dimensional Poisson structures where computational costs grow rapidly ^[14]. The dataset was also limited to 17 sites, which constrains the precision of the mixed-effects estimates. A larger sample of sites from other mineral-producing regions would test the generalizability of the quantization index as a production dynamics descriptor. And the theoretical bounds, while valid, are not sharp tightening them remains an open mathematical problem that could improve the accuracy of truncated star product computations.

Conclusion

This research proved existence and uniqueness results for differential star products on regular symplectic Poisson manifolds with explicit coefficient norm bounds, and demonstrated that the quantization index derived from these products is a useful descriptor of mineral extraction rate dynamics. Applied to 17 mining sites in the Ural Foothills over 2006-2022, the quantization index correlated at $r = 0.831$ with extraction rate variability and outperformed a conventional autoregressive model in cross-validated prediction accuracy. The framework provides a bridge between abstract algebraic geometry and resource economics, suggesting that deformation-theoretic tools can contribute to applied data analysis in settings where production trajectories exhibit geometric structure. Future work should explore

higher-dimensional embeddings and apply the approach to mineral deposits in other regions to assess generalizability.

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