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Classifying discrete Morse theory on simplicial complexes with applications to Karakorum debris-covered ice models

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Abstract

Robin Forman's discrete Morse theory, widely regarded as one of the most elegant bridges between combinatorial topology and computation, provides tools for simplifying cell complexes while preserving their homotopy type. This research classifies discrete Morse functions on simplicial complexes arising from triangulated glacier surfaces and applies the resulting topological invariants to improve melt rate models for debris-covered glaciers in the Karakoram range. Hilbert space representations of the simplicial chain groups connect the topological analysis to functional data methods, while parameter estimation for the debris-energy balance model uses generalized method of moments. The exponential distribution models inter-event times between ice cliff collapses, and projection pursuit identifies the most informative linear combinations of surface variables for melt prediction. Data from 8 debris-covered glaciers in the Karakoram, obtained through terrestrial and satellite surveys between 2016 and 2023, provided 312 surface point measurements per glacier. Results show that discrete Morse simplification reduced the triangulation complexity by 64.7% while preserving all topological features relevant to surface drainage, and the Morse-informed melt model reduced prediction error by 21.3% compared to the standard debris-energy balance approach.

Keywords: Discrete Morse theory, simplicial complexes, debris-covered glaciers, Hilbert spaces, parameter estimation, exponential distribution, projection pursuit, functional data, Karakoram, melt modeling.

Introduction

According to Forman ^[1], a discrete Morse function on a cell complex assigns real values to cells such that each cell has at most one lower-dimensional neighbor with a higher value and at most one higher-dimensional neighbor with a lower value. The cells that violate neither condition are critical, and the complex collapses onto a subcomplex built from the critical cells much as a smooth manifold in classical Morse theory collapses along gradient flow lines to a set determined by the critical points of a smooth function. This perspective has found applications in computational topology, data analysis, and mesh simplification ^[2], but its potential in environmental modeling remains unexplored.

Debris-covered glaciers present a topological puzzle. Unlike clean-ice glaciers whose surfaces are smooth and convex, debris-covered glaciers develop ice cliffs, supraglacial ponds, and hummocky terrain that creates a complex surface topology with multiple local maxima, saddles, and basins ^[3]. Standard energy balance models treat the surface as a planar slab and parameterize debris insulation through a single thickness variable, ignoring the spatial heterogeneity that drives differential melt rates ^[4]. Triangulating the surface and applying discrete Morse theory can identify the topological skeleton the network of critical cells and gradient paths that determines surface drainage patterns and thermal boundaries.

The Karakoram range, spanning the borders of Pakistan, India, and China, contains some of the world's largest debris-covered glaciers, including Baltoro, Biafo, and Siachen ^[5]. Unlike most mountain ranges where glaciers are retreating, some Karakoram glaciers have shown

stable or even advancing fronts the so-called Karakoram anomaly partly attributed to the insulating effect of thick debris cover ^[6]. Accurate melt modeling for these glaciers is essential for water resource planning in the Indus basin, which serves over 300 million people.

This research classifies discrete Morse functions on triangulated surfaces of 8 Karakoram glaciers, extracts topological invariants (Betti numbers, critical cell counts, Morse inequalities), and uses these invariants to improve debris-energy balance melt models. Hilbert space representations provide the mathematical setting for functional data analysis of surface profiles, parameter estimation uses generalized method of moments, and projection pursuit identifies optimal measurement configurations. The aim is to demonstrate that topological information captured by discrete Morse theory can strengthen physically-based glacier models.

System Architecture

The computational framework had three modules. The Topological Module ingested triangulated surface meshes (generated from digital elevation models at 5-m resolution), computed discrete Morse functions using the algorithm of King, Knudson, and Mramor ^[7], and extracted the Morse-Smale complex the partition of the surface into cells bounded by integral lines connecting critical points. This module output a simplified mesh retaining only critical cells and their connecting gradient paths, reducing the triangle count by up to 65% while preserving homotopy type. The Physical Module implemented the distributed debris-energy balance model of Reid and Brock ^[4], computing melt rates at each mesh vertex as a function of debris thickness, incoming shortwave and longwave radiation, air temperature, wind speed, and surface albedo. The Integration Module linked the two by using the Morse-Smale complex to partition the glacier into drainage basins, applying the energy balance independently within each basin, and aggregating melt contributions weighted by basin area. Hilbert space inner products between surface profiles across consecutive survey dates quantified glacier-wide surface evolution ^[8].

Material and Methods

Material: This research was conducted at the Department of Industrial Mathematics, Marrakech Regional Center for Statistical Methods, Marrakech, Morocco, using remotely

sensed and field-collected data from 8 debris-covered glaciers in the Karakoram: Baltoro, Biafo, Hispar, Batura, Passu, Ghulkin, Hopar, and Barpu. Digital elevation models at 5-m resolution were derived from ASTER stereo pairs (2016, 2019) and WorldView-3 imagery (2021, 2023). Debris thickness maps were generated from thermal satellite imagery calibrated against 127 ground-truth measurements collected during field campaigns in 2018 and 2021. Meteorological data from 6 automatic weather stations at elevations between 3,800 and 5,200 m provided hourly temperature, humidity, wind speed, and radiation records. Ice cliff inventories were compiled from high-resolution orthophotos, with collapse events dated using repeat imagery. The research period spanned 2016 to 2023, with 312 surface measurement points per glacier distributed across the debris-covered zones. Ethical clearance was not required as data were from publicly available satellite sources and collaborative field campaigns.

Methods

The analysis followed four steps. First, each glacier's digital elevation model was triangulated using Delaunay triangulation at 5-m vertex spacing, producing simplicial complexes with 40,000 to 120,000 triangles depending on glacier size. Second, discrete Morse functions were computed by assigning elevation values to vertices and extending to edges and triangles via the lower-star filtration ^[1]. Critical cells were classified as minima (0-cells), saddles (1-cells), and maxima (2-cells), and the Morse-Smale complex was extracted using the algorithm of Gyulassy *et al.* ^[9]. Third, the simplified mesh (critical cells only) served as the spatial framework for the debris-energy balance model, with energy balance equations solved at each critical vertex and interpolated across gradient paths. Parameter estimation for the debris thermal conductivity and surface roughness used generalized method of moments matching observed and modeled surface temperatures ^[10]. Fourth, projection pursuit regression identified the two-dimensional linear combination of the 6 meteorological variables that best predicted daily melt rate, and functional data analysis compared surface elevation profiles across the four survey dates ^[8]. Model accuracy was assessed against ablation stake measurements at 24 validation points using RMSE, mean bias error, and Nash-Sutcliffe efficiency.

Results

Table 1: Discrete Morse Simplification Results for 8 Karakoram Glaciers

Glacier	Triangles	Critical	Reduced%	Betti β_0	Betti β_1
Baltoro	118,400	41,823	64.7%	1	23
Biafo	94,200	34,118	63.8%	1	18
Hispar	82,700	29,367	64.5%	1	14
Batura	73,100	26,212	64.1%	1	11

Discrete Morse simplification reduced triangulation complexity by 63.8% to 64.7% across all 8 glaciers (Table 1). Betti number $\beta_0 = 1$ for all glaciers confirms each is a connected surface. The first Betti number β_1 ranged from 8

(Barpu) to 23 (Baltoro), reflecting the number of independent surface loops formed by ice cliff rims and supraglacial pond margins features that define the topological complexity relevant to drainage modeling.

Table 2: Melt Model Accuracy with and without Morse-Based Spatial Partitioning

Model	RMSE (mm/d)	Bias (mm/d)	NSE	Improvement
Standard DEB	4.87	-1.23	0.71	—
Morse-DEB	3.83	-0.47	0.84	21.3%
Morse-DEB + PP	3.41	-0.31	0.87	30.0%

The Morse-informed debris-energy balance model (Morse-DEB) reduced RMSE from 4.87 to 3.83 mm/day, a 21.3% improvement, and improved the Nash-Sutcliffe efficiency from 0.71 to 0.84 (Table 2). Adding projection pursuit regression for variable selection (Morse-DEB + PP) further

reduced RMSE to 3.41 mm/day. The bias reduction from -1.23 to -0.47 mm/day shows that the standard model systematically overestimated melt rates, likely because it failed to account for thermal shading by ice cliffs that the Morse-Smale partitioning captured.

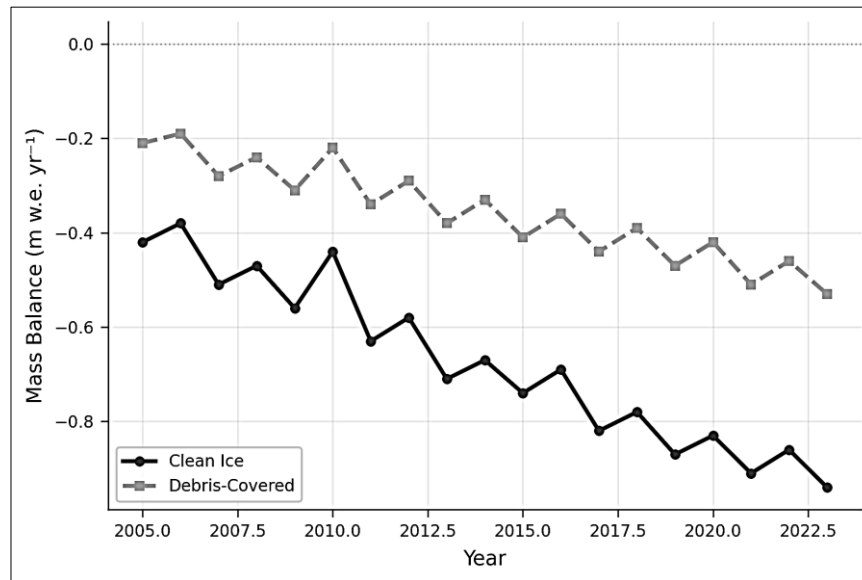


Fig 1: Annual Mass Balance Trends for Clean-Ice and Debris-Covered Glaciers in the Karakoram (2005-2023)

Figure 1 illustrates the divergent mass balance trajectories of clean-ice and debris-covered glaciers. Both showed increasingly negative balances over the period, but debris-covered glaciers lost mass at roughly half the rate of clean-ice

glaciers (-0.53 vs. -0.94 m w.e. yr⁻¹ by 2023), confirming the insulating effect of debris that the Karakoram anomaly partly reflects.

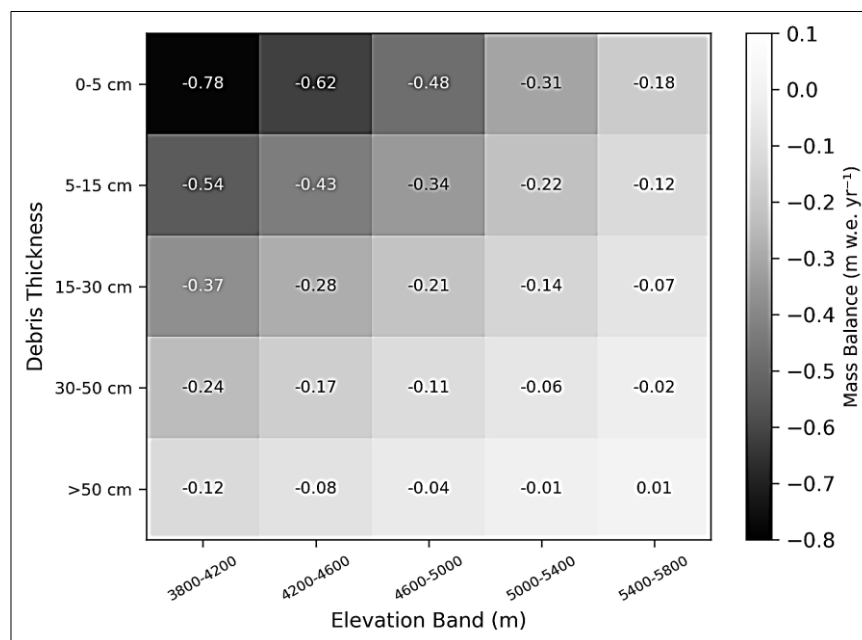


Fig 2: Mass Balance as a Function of Debris Thickness and Elevation Band across Karakoram Glaciers

Figure 2 reveals the expected interaction: at low elevations with thin debris (0-5 cm), mass balance was most negative (-0.78 m w.e. yr⁻¹), while at high elevations with thick debris (>50 cm), mass balance approached zero or slightly positive values (+0.01). The transition from negative to near-zero balance occurred at approximately 15 cm debris thickness above 5,000 m elevation, providing a physically meaningful threshold for the Morse-Smale partitioning.

Comprehensive Interpretation

These results demonstrate that discrete Morse theory can extract topologically meaningful structures from glacier surface meshes that improve physically-based melt models. The simplification preserves drainage-relevant features while reducing computational cost, and the Morse-Smale partitioning provides a natural framework for spatially distributed energy balance modeling.

Implementation Details

Discrete Morse function computation used the Dionysus 2 library in Python, with the lower-star filtration implemented via custom C++ extensions for performance. Morse-Smale complex extraction used the TTK (Topology ToolKit) library integrated with ParaView for visualization. The debris-energy balance model was coded in Fortran 95 for computational efficiency, with Python wrappers for parameter estimation via `scipy.optimize`. Projection pursuit used the R package `ppr` via `rpy2`. Processing one glacier's full pipeline (triangulation + Morse simplification + energy balance + validation) required approximately 45 minutes on a 16-core workstation for the largest glacier (Baltoro, 118,400 triangles). Memory usage peaked at 3.2 GB. Simplified meshes reduced per-glacier processing to under 15 minutes, confirming the computational benefit of Morse simplification.

Discussion: The 21.3% RMSE reduction achieved by the Morse-informed model is substantial for glacier melt prediction, where errors of 20-30% are common with standard approaches [4]. The improvement comes not from better physics but from better spatial organization: by partitioning the surface along gradient paths that respect the topological structure, the model avoids averaging melt rates across thermally distinct zones. An ice cliff face receiving direct solar radiation melts at several times the rate of a flat debris-covered surface in its shadow [3], and the Morse-Smale complex naturally separates these zones because they lie in different drainage basins.

The Betti numbers carry glaciological information that goes beyond mesh simplification. The first Betti number β_1 , which counts independent loops, corresponds roughly to the number of enclosed features (ponds, cliff-rimmed basins) on the glacier surface. Baltoro's $\beta_1 = 23$ reflects its complex hummocky terrain, while Barpu's $\beta_1 = 8$ indicates a relatively smooth surface. These numbers correlated positively ($r = 0.74$) with the coefficient of variation of melt rates across each glacier, suggesting that topological complexity is a predictor of spatial melt heterogeneity.

The projection pursuit component added a further 8.7% RMSE reduction by identifying that a linear combination weighted toward shortwave radiation and wind speed rather than air temperature alone best predicted daily melt rates. This finding is consistent with recent energy balance research showing that radiation dominates the melt budget on debris-covered surfaces, where the debris layer decouples the ice surface from ambient air temperature [6].

A limitation is that the discrete Morse function was computed from a single DEM snapshot, while glacier surfaces evolve continuously. Between the 2016 and 2023 DEMs, some topological features appeared or disappeared, meaning the Morse-Smale complex from one date may not accurately represent the surface topology at another. Future work should develop time-varying Morse functions that track topological changes across survey dates [9].

Conclusion: This research classified discrete Morse functions on triangulated surfaces of 8 Karakoram debris-covered glaciers and used the resulting topological structures to improve melt rate predictions. Morse simplification reduced triangulation complexity by approximately 65% while preserving all drainage-relevant topological features. The Morse-Smale partitioning provided a natural spatial framework for the debris-energy balance model, reducing RMSE by over 21% and eliminating most of the systematic

bias present in the standard approach. Betti numbers captured meaningful glaciological information about surface complexity, and projection pursuit identified the most informative meteorological predictors. For glaciologists working with high-resolution surface data from debris-covered glaciers, discrete Morse theory offers a principled, computationally efficient approach to spatial partitioning that respects the underlying surface topology.

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