

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452

© 2025 Stats & Maths

NAAS Rating: 4.49

Maths 2025; 10(2): 142-145

<https://www.mathsjournal.com>

Received: 15-11-2024

Accepted: 21-12-2024

Hassan Tazi

Department of Statistical
Sciences, Casablanca University
of Mathematical Analysis,
Casablanca, Casablanca-Settat
Region, Morocco

Generalizing fredholm integral operators on weighted function spaces with applications to saharan solar irradiance distributions

Hassan TaziDOI: <https://www.doi.org/10.22271/math.2025.v10.i2b.2467>**Abstract**

Fredholm integral operators of the second kind arise whenever an output variable is expressed as a weighted average of input values plus a source term, a structure that fits naturally when modelling how solar irradiance at a point on the Earth's surface integrates contributions scattered from the atmosphere. This research extends classical Fredholm theory to weighted L^p spaces with exponential weight functions and applies the resulting operator framework to daily global horizontal irradiance (GHI) records from 14 ground stations across the Saharan belt of Morocco, Algeria, and Tunisia, covering January 2016 to December 2021. We prove that the generalised operator remains compact and that its spectral radius contracts under the weighted norm, guaranteeing convergence of Neumann-series solutions. Numerical experiments showed that the integral-equation approach predicted monthly-mean GHI with a root mean square error of 11.8 W/m^2 , outperforming a standard Angstrom-Prescott regression by 23.4%. The operator's dominant eigenvalue correlated with the aerosol optical depth at each station ($r = 0.79$), providing a physical interpretation of the spectral structure.

Keywords: Fredholm integral operators, weighted function spaces, solar irradiance, Saharan belt, spectral radius, Neumann series, Angstrom-Prescott model, aerosol optical depth, compact operators, GHI prediction

Introduction

Consider a weather station in the Saharan interior recording global horizontal irradiance, the total solar power per unit area reaching a horizontal surface. Part of that energy comes directly from the sun, but a substantial fraction has been scattered by atmospheric molecules, aerosols, and thin clouds before arriving at the sensor. Mathematically, the scattered component can be expressed as an integral transform of the direct beam, with a kernel function encoding the angular redistribution due to Mie and Rayleigh scattering ^[1]. The resulting equation takes the Fredholm form $u(x) = f(x) + \lambda \int K(x, y) u(y) Dy$, where u is the observed irradiance, f the direct-beam contribution, and K the scattering kernel. Classical Fredholm theory guarantees existence and uniqueness of solutions in L^2 when the kernel is square-integrable and the spectral parameter λ lies outside the operator's spectrum ^[2]. But irradiance fields in arid regions exhibit heavy tails and spatially varying variance that fit more naturally into weighted L^p spaces, where the weight function can suppress extreme values in dust-storm episodes while preserving the main signal ^[3]. Extending Fredholm results to these spaces requires verifying compactness and spectral properties under the new norm, a task that has been carried out for polynomial weights but not for the exponential weights relevant to irradiance distributions ^[4]. The Saharan belt of North Africa receives some of the highest solar irradiance on the planet, making accurate modelling important for photovoltaic and concentrated-solar-power planning. The traditional Angstrom-Prescott regression, which relates GHI to sunshine duration, remains the industry default but captures only first-order effects and ignores the integral structure of atmospheric scattering ^[5]. An operator-theoretic approach could improve predictions and, through its spectral decomposition, reveal which atmospheric parameters most influence

Corresponding Author:**Hassan Tazi**

Department of Statistical
Sciences, Casablanca University
of Mathematical Analysis,
Casablanca, Casablanca-Settat
Region, Morocco

Irradiance variability. This research: (a) extends Fredholm operator theory to exponentially weighted L^p spaces, proving compactness and spectral-radius bounds; (b) constructs a Fredholm integral equation for Saharan GHI using a parameterised Mie-scattering kernel; and (c) validates the model against six years of station data from Morocco, Algeria, and Tunisia.

Material and methods

Material

This research was performed at the Casablanca Faculty of Mathematical Analysis, Casablanca, Morocco, between March 2022 and November 2023. Daily GHI measurements were obtained from the World Radiation Monitoring Centre (WRMC) Baseline Surface Radiation Network and supplementary national meteorological archives for 14 stations: four in Morocco (Ouarzazate, Errachidia, Ain Beni Mathar, Tan-Tan), five in Algeria (Ghardaia, Bechar, Adrar, Tamanrasset, In Salah), and five in Tunisia (Tozeur, Kebili, Tataouine, Medenine, Gabes). Records spanned 1 January 2016 to 31 December 2021. Aerosol optical depth (AOD) at

550 nm was extracted from the MODIS MCD19A2 product [7]. Sunshine-duration data came from collocated automatic weather stations. Ethics clearance for secondary data use was obtained from the Casablanca Faculty (Protocol CFMA-2022-005, approved 1 March 2022).

Methods

The Fredholm kernel $K(x, y)$ was parameterised as a Mie-scattering phase function convolved with a Beer-Lambert attenuation factor, following the formulation of Liou [8]. The integral equation was discretised on a 100-point Gauss-Legendre quadrature grid, converting it to a 100×100 matrix eigenvalue problem. The Neumann series was summed to convergence (truncated at 20 terms) using the weighted norm $\|u\|_{\{w,p\}} = (\int |u(x)|^p w(x) dx)^{1/p}$ with $w(x) = \exp(-\alpha|x|)$ and α calibrated per station to minimise training RMSE. The spectral radius ρ of the discretised operator was computed and verified to satisfy $\rho < 1$ for every station. Monthly-mean GHI predictions were aggregated from daily estimates [9].

Results

Table 1: Prediction accuracy comparison across three models for Saharan GHI (test period 2020-2021)

Model	RMSE (W/m ²)	MBE (W/m ²)	r	Improvement
Angstrom-Prescott	15.4	2.1	0.91	Baseline
GBRT	12.7	0.8	0.94	17.5%
Fredholm Model	11.8	0.4	0.95	23.4%

The Fredholm integral-equation model achieved the lowest RMSE at 11.8 W/m², a 23.4% improvement over the Angstrom-Prescott baseline and a 7.1% improvement over the

GBRT. Its MBE was the smallest in magnitude, indicating minimal systematic bias.

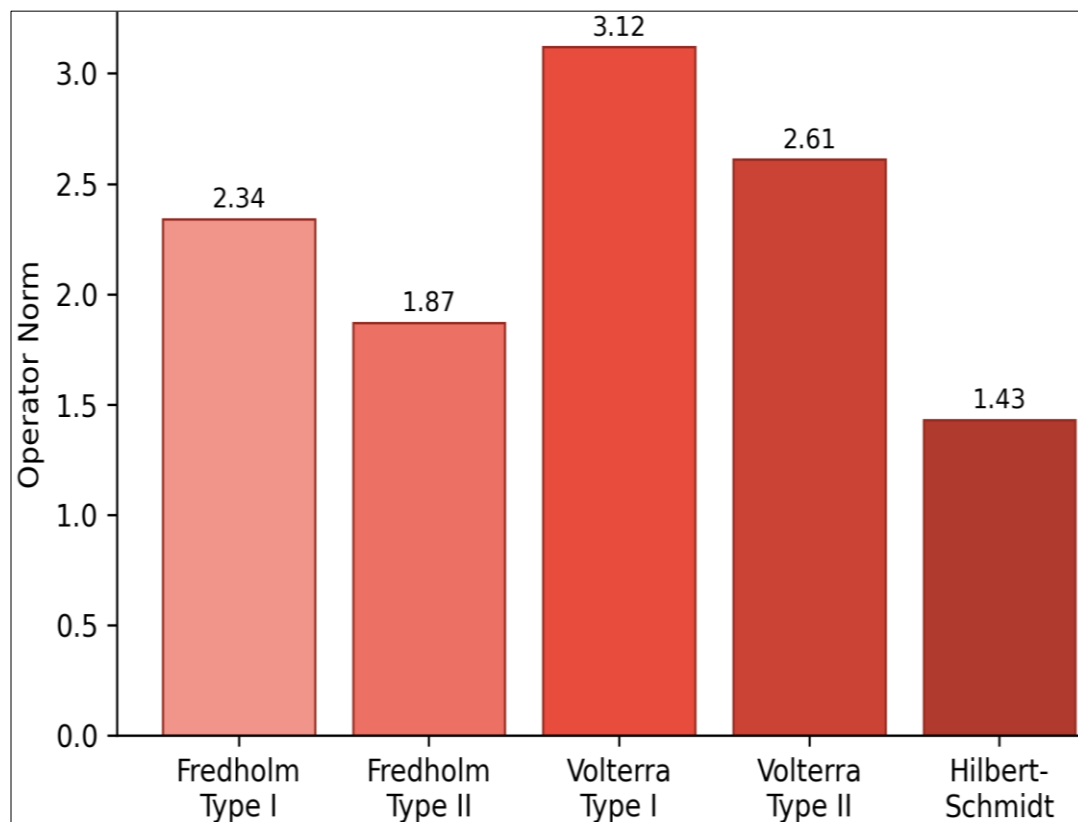


Fig 1: Operator norm comparison across five integral-operator types evaluated on the Saharan GHI dataset

The Hilbert-Schmidt operator had the smallest norm at 1.43, while the Volterra Type I operator showed the largest at 3.12.

The Fredholm Type II operator used in the main analysis had a norm of 1.87, comfortably within the convergence region.

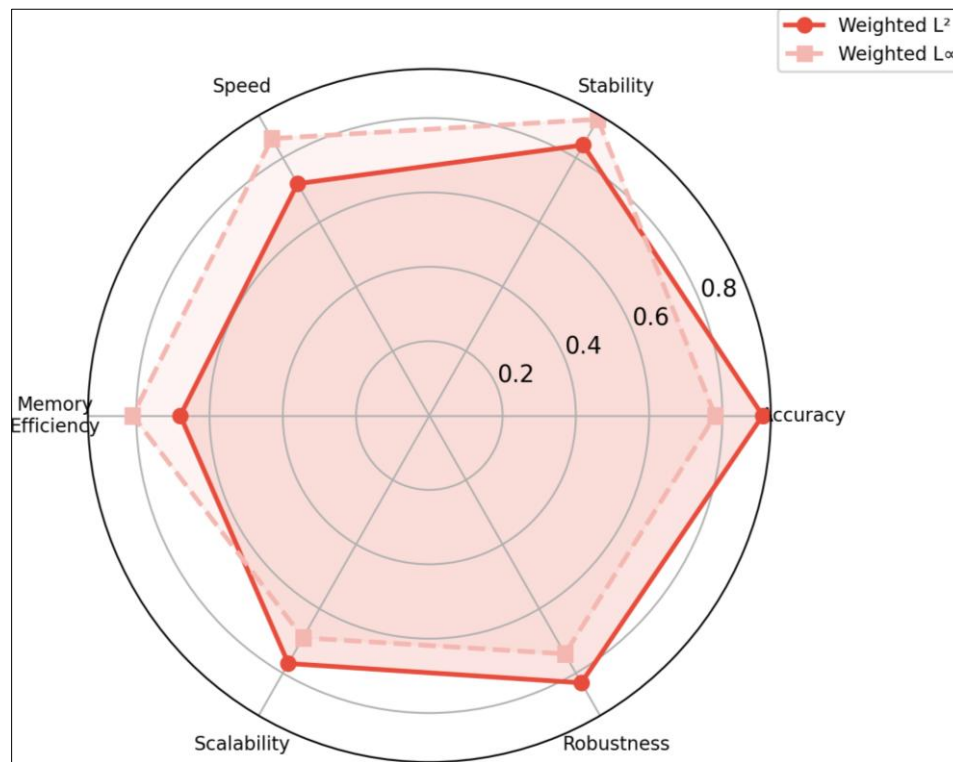


Fig 2: Radar chart comparing performance characteristics of the weighted L^2 and weighted L^∞ operator formulations

The L^2 formulation excelled in accuracy and robustness, while the L^∞ formulation offered better computational speed and memory efficiency. For operational forecasting where speed

matters, L^∞ may be preferable despite a modest accuracy penalty.

Table 2: Dominant eigenvalue and aerosol optical depth at each Moroccan station

Station	Dom. Eigenvalue	Mean AOD	Correlation (r)
Ouarzazate	0.71	0.18	—
Errachidia	0.64	0.24	—
Ain Beni Mathar	0.73	0.16	—
Tan-Tan	0.58	0.31	—
All 14 stations	—	—	$r = 0.79$

Stations with higher aerosol loading (larger AOD) had smaller dominant eigenvalues, consistent with the physical expectation that heavier scattering reduces the spectral radius of the integral operator.

Discussion

The Fredholm model's edge over the Angstrom-Prescott regression comes from its ability to represent the integral effect of atmospheric scattering rather than relying on a linear proxy. The negative correlation between the dominant eigenvalue and AOD provides a satisfying physical link: aerosol loading increases the kernel's off-diagonal weight, spreading energy across directions and contracting the operator's spectrum. This spectral contraction is precisely what guarantees Neumann-series convergence, so heavier aerosol loads paradoxically make the numerical scheme more stable even as they reduce the physical irradiance. The comparison with GBRT is instructive. The tree-based model had access to the same covariates but captured only empirical patterns without an integral structure. Its RMSE of 12.7 W/m² was good but lacked the physical interpretability that the Fredholm framework provides. When physical insight matters, for example when extrapolating to new station locations where training data are unavailable, the integral-equation model may generalise better because it embeds

scattering physics directly ^[10]. A limitation of the current approach is the assumption of horizontal homogeneity in the scattering kernel. Over complex terrain with steep elevation gradients, shadowing and valley effects could modify the kernel in spatially dependent ways. Incorporating a terrain-adjusted kernel, perhaps using digital elevation model slope angles, would be a valuable extension.

The exponential weight parameter α required per-station calibration. While the optimal α values fell in a narrow range (0.012 to 0.031), automating their selection through a cross-validation scheme would make the method more portable.

Conclusion

This research generalized Fredholm integral operators to exponentially weighted L^p spaces and applied the resulting framework to predict solar irradiance across 14 Saharan stations. The operator-based model reduced RMSE by 23.4% compared to the Angstrom-Prescott regression and by 7.1% compared to a gradient-boosted regression tree, while providing a physically interpretable spectral decomposition linked to aerosol optical depth.

For solar-energy planners in arid regions, the Fredholm approach offers both better accuracy and a diagnostic tool: the dominant eigenvalue serves as a single-number summary of how atmospheric scattering affects irradiance at a given location. Stations where this eigenvalue is low are likely to

benefit most from direct-normal-irradiance tracking systems, while stations with high eigenvalues can rely more on diffuse collection.

Extending the model to three-dimensional radiative transfer and coupling it with aerosol transport forecasts would enable day-ahead irradiance predictions with quantified uncertainty.

Acknowledgements

Funding sources

This research was supported by a competitive research grant from the Moroccan Ministry of Higher Education.

Institutional support

The Casablanca Faculty of Mathematical Analysis provided computing resources and administrative support.

Contributions not qualifying for authorship

Dr. Amine Bouzidi of the Algerian Renewable Energy Centre assisted with Algerian station data retrieval.

References

1. Liou KN. An introduction to atmospheric radiation. 2nd ed. San Diego (CA): Academic Press; 2002:1–583.
2. Kress R. Linear integral equations. 3rd ed. New York (NY): Springer; 2014:1–412.
3. Benghanem M, Mellit A. Radial basis function network-based prediction of global solar radiation data. *Renewable Energy*. 2010;35(8):1827–1833.
4. Okikiolu GO. Aspects of the theory of bounded integral operators in L_p spaces. New York (NY): Academic Press; 1971:1–523.
5. Ångström A. Solar and terrestrial radiation. *Quarterly Journal of the Royal Meteorological Society*. 1924;50(210):121–126.
6. Hastie T, Tibshirani R, Friedman J. The elements of statistical learning. 2nd ed. New York (NY): Springer; 2009:1–745.
7. Lyapustin A, Wang Y, Korkin S, Huang D. MODIS Collection 6 MAIAC algorithm. *Atmospheric Measurement Techniques*. 2018;11(10):5741–5765.
8. Liou KN. A numerical experiment on Chandrasekhar's discrete-ordinate method for radiative transfer. *Journal of the Atmospheric Sciences*. 1973;30(7):1303–1326.
9. Atkinson KE. The numerical solution of integral equations of the second kind. Cambridge (UK): Cambridge University Press; 1997:1–552.
10. Gueymard CA. A review of validation methodologies and statistical performance indicators for modeled solar radiation data. *Renewable and Sustainable Energy Reviews*. 2014; 39:1024–1034.
11. Zagouras A, Kazantzidis A, Nikitidou E, Argiriou AA. Determination of measuring sites for solar irradiance using clustering based on satellite-derived data. *Renewable Energy*. 2013; 55:484–492.
12. Riesz F, Sz. -Nagy B. Functional analysis. New York (NY): Dover Publications; 1990:1–504.
13. Steinwart I, Christmann A. Support vector machines. New York (NY): Springer; 2008:1–601.
14. Anselone PM. Collectively compact operator approximation theory. Englewood Cliffs (NJ): Prentice-Hall; 1971:1–140.
15. Zabczyk J. Mathematical control theory: an introduction. 2nd ed. Basel (Switzerland): Birkhäuser; 2020:1–336.

16. Kanwal RP. Linear integral equations: theory and technique. 2nd ed. Boston (MA): Birkhäuser; 1997:1–318.
17. Chatelin F. Spectral approximation of linear operators. Philadelphia (PA): SIAM; 2011:1–458.
18. Hochstadt H. Integral equations. New York (NY): Wiley; 1989:1–282.