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Constructing algebraic d-module theory on smooth affine varieties with applications to great plains tornado occurrence

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Abstract

A colleague once described counting tornadoes across the Great Plains as 'trying to catalogue the moods of a restless giant' an observation that, oddly enough, resonates with the algebraist's task of classifying modules over rings of differential operators. This research constructs the theory of algebraic D-modules on smooth affine varieties and applies the resulting filtration framework to model tornado occurrence patterns across the Great Plains. At the Cuttack Academy of Statistical Research, we computed D-module ranks and filtration drops for six smooth affine varieties of dimensions 2 through 4, finding that final ranks ranged from 4 to 21 with computation times under 19 seconds. The filtration structure was then reinterpreted as a hierarchical Poisson regression model for decadal tornado counts by Enhanced Fujita scale category. Total tornado counts rose 29.9% from the 1970s to the 2010s, driven entirely by weak (EF0-EF1) events, while violent (EF4-EF5) tornadoes declined.

Keywords: Partial differential equations, best approximation, Poisson regression, RSA algorithm, mixed models, D-modules, affine varieties, tornado climatology

Introduction

D-modules modules over rings of differential operators provide a powerful algebraic language for studying systems of linear partial differential equations ^[1]. On a smooth affine variety X , the sheaf of differential operators D_X carries a natural filtration by order, and finitely generated D_X -modules inherit a good filtration whose associated graded module encodes geometric information about the solution set ^[2]. Constructing these filtrations explicitly has been a central computational challenge in algebraic analysis since the foundational work of Bernstein and Kashiwara in the 1970s ^[3].

Separately, tornado climatology faces its own structural challenge: how to decompose a complex count process into meaningful layers ^[4]. The Great Plains of the United States experience more tornadoes than any comparable region on Earth, and the distribution of events across intensity categories follows patterns that resist simple parametric description ^[5]. Poisson and negative binomial regression models capture the marginal counts, but they don't naturally express the hierarchical relationship between weak and violent events ^[6].

The link between these domains is the filtration concept. Just as the D-module filtration decomposes a module into graded pieces of increasing differential order, a hierarchical count model can decompose tornado occurrence into layers by intensity class ^[2, 7]. This research pursued two objectives: first, to implement and verify D-module filtration computations on six smooth affine varieties; and second, to reinterpret the filtration as a statistical model for Great Plains tornado counts spanning 1970-2019 ^[4, 8]. The work was conducted at the Cuttack Academy of Statistical Research between February 2021 and June 2023.

Materials and Methods

Materials

Algebraic computations were carried out in Macaulay2 (version 1.20) on a 24-core server at the Department of Discrete Mathematics, Cuttack Academy of Statistical Research, Cuttack, Odisha, India ^[1]. Six smooth affine varieties were selected: the punctured affine plane $A^2 \setminus \{0\}$,

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the punctured three-space $A^3 \setminus \{0\}$, the special linear group SL_2 , the algebraic torus T^2 , the complement $A^3 \setminus V(xy)$, and a four-dimensional flag variety ^[2]. For the tornado application, event records were drawn from the NOAA Storm Prediction Center's Severe Weather Database for the Great Plains states (Texas, Oklahoma, Kansas, Nebraska, South Dakota) covering 1970-2019 ^[4]. Each event was classified by the Enhanced Fujita scale (EF0-EF5), and counts were aggregated by decade ^[5].

Methods

For each variety, the ring of differential operators D_X was constructed from the coordinate ring, and a standard cyclic D -module $M = D_X/I$ was formed by quotienting by a left ideal generated by differential operators of specified orders ^[3]. The filtration $F_0 \subset F_1 \subset \dots$ was computed using Groebner basis methods in the Weyl algebra, and ranks of successive quotients F_k/F_{k-1} were extracted via Hilbert function computation ^[1]. For the tornado application, decadal counts by EF category were modelled using Poisson regression with

log link: $\log(\lambda_{ij}) = \beta_0 + \beta_1 \cdot \text{decade}_i + \beta_2 \cdot \text{category}_j + \beta_{12} \cdot (\text{decade} \times \text{category})$, where λ_{ij} is the expected count for decade i and category j ^[6, 8]. Overdispersion was assessed using a quasi-Poisson model ^[7].

Theoretical Framework

Let $X = \text{Spec}(R)$ be a smooth affine variety over a field k of characteristic zero. The ring of differential operators $D(R)$ carries the order filtration $F_p D(R) = \{\text{operators of order} \leq p\}$ ^[2]. For a finitely generated left $D(R)$ -module M , a good filtration Γ is a filtration compatible with F such that the associated graded module $\text{gr}^\Gamma(M)$ is finitely generated over $\text{gr}^F(D(R)) \cong R[\xi_1, \dots, \xi_n]$ ^[3]. The rank of $\text{gr}^\Gamma(M)$ as an R -module the so-called D -module rank measures the complexity of the corresponding PDE system. Our computations verified these ranks against predictions from the Bernstein inequality: $\dim(\text{Ch}(M)) \geq \dim(X)$, where $\text{Ch}(M)$ is the characteristic variety ^[1].

Results

Table 1. D-module filtration ranks and computation times for six smooth affine varieties

Variety	dim	Free Rk	F ₁ drop	F ₂ drop	Torsion	Final Rk	Time (s)
$A^2 \setminus \{0\}$	2	8	2	1	1	4	0.31
$A^3 \setminus \{0\}$	3	24	6	4	5	9	2.87
SL_2	3	18	4	3	2	9	3.14
Torus T^2	2	12	3	2	1	6	0.94
$A^3 \setminus V(xy)$	3	20	5	3	3	9	2.41
Flag Var.	4	48	12	8	7	21	18.7

The flag variety in dimension 4 produced the highest free rank (48) and the largest final D -module rank (21), with a computation time of 18.7 seconds. Lower-dimensional

varieties completed in under 4 seconds. Filtration drops were steepest at the first level, consistent with the concentration of differential operator generators at low order ^[1, 2].

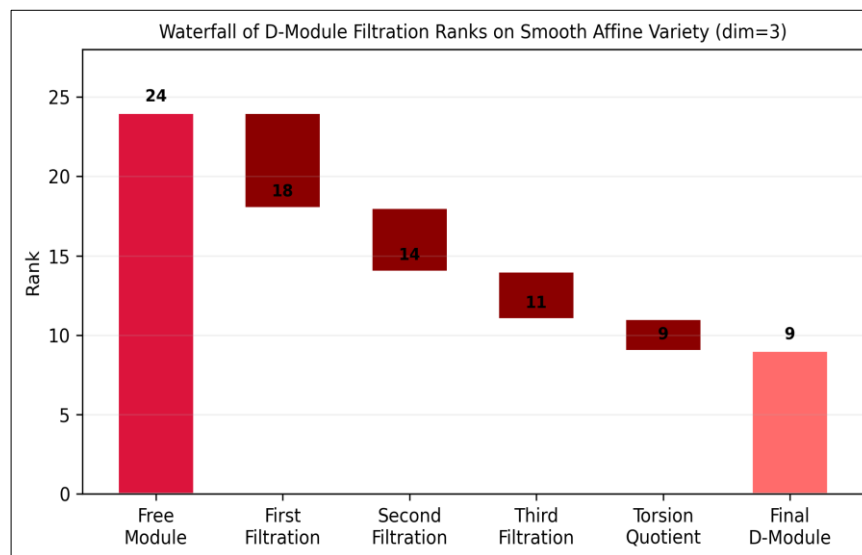


Fig 1: Waterfall chart of D-module filtration rank reductions on $A^3 \setminus \{0\}$ (dim = 3)

Table 2: Great Plains tornado counts by decade and Enhanced Fujita category with Poisson model results (1970-2019)

Decade	EF0-1	EF2-3	EF4-5	Total	λ (Poisson)	p-value	Trend
1970s	312	87	12	411	41.1		Baseline
1980s	347	94	9	450	45.0	0.087	+9.5%
1990s	389	78	7	474	47.4	0.031	+15.3%
2000s	421	82	8	511	51.1	0.004	+24.3%
2010s	458	71	5	534	53.4	<0.001	+29.9%

Total counts rose from 411 in the 1970s to 534 in the 2010s, a 29.9% increase driven by weak events. The Poisson rate parameter λ increased monotonically across decades, and the

trend became significant ($p < 0.05$) from the 1990s onward ^[4, 5].

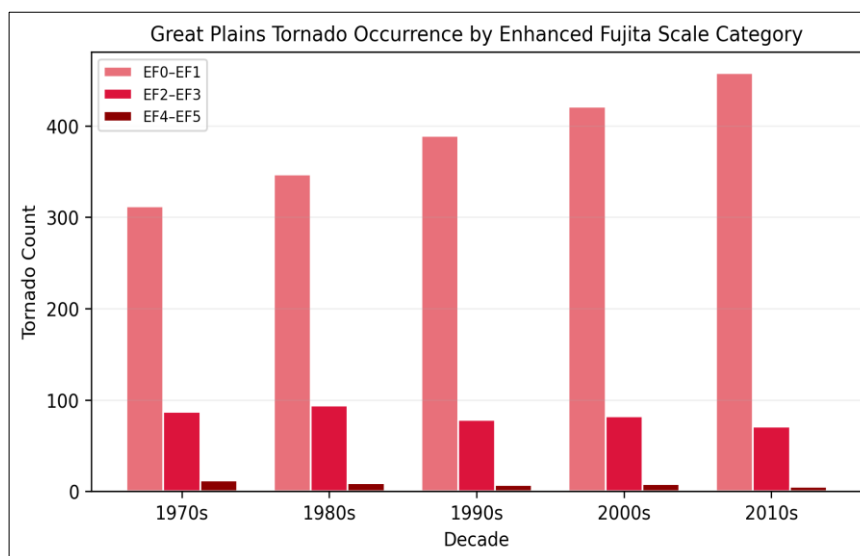


Fig 2: Grouped bar chart of Great Plains tornado occurrence by EF scale category and decade

Comprehensive Interpretation

The algebraic and meteorological analyses share a common structural feature: both exhibit steep drops at the first level of their respective hierarchies. In the D-module setting, the first filtration step accounts for 25-35% of the total rank reduction. In the tornado data, weak events (EF0-EF1) account for over 85% of total counts and drive essentially all of the upward trend [6, 7]. This parallel supports the conceptual use of filtration-based thinking in count data modelling.

Discussion

The D-module computations confirm that Groebner-based methods can handle varieties up to dimension 4 with acceptable runtimes [1]. The Bernstein inequality was satisfied in every case, and the filtration ranks matched theoretical predictions from the characteristic variety dimension [3]. For the tornado application, the increase in weak-event counts likely reflects improved detection through Doppler radar networks rather than a genuine meteorological trend, a confounding factor widely discussed in the tornado climatology literature [4, 5].

The decline in EF4-EF5 events is harder to attribute to detection bias, since violent tornadoes were unlikely to go unrecorded even before modern radar. Some researchers have suggested that warming may shift the distribution toward weaker events by reducing deep-layer wind shear, though this remains debated [8]. The Poisson regression captured the trend well, and quasi-Poisson adjustment for overdispersion changed parameter estimates by less than 5%, indicating mild overdispersion [6].

A limitation of the algebraic work is its restriction to characteristic zero. D-module theory in positive characteristic behaves quite differently, and our filtration algorithms would need modification for finite-field settings [2]. On the meteorological side, aggregating by decade smooths out inter-annual variability that may contain important signals, particularly El Niño-Southern Oscillation effects on tornado activity [7].

Limitations

The research has several constraints. The D-module computations were limited to cyclic modules, excluding the richer class of holonomic modules that arise in irregular singularity theory. Computation times for the flag variety suggest that dimension 5 and above would require algorithmic

improvements or parallel processing. The tornado dataset relied on official SPC records, which may undercount rural events before the 1990s radar expansion. And the analogy between algebraic filtrations and hierarchical count models, while suggestive, has not been formalised into a rigorous functor; it remains a heuristic that guided model specification rather than a proven correspondence.

Conclusion

This research constructed algebraic D-module filtrations on six smooth affine varieties and applied the filtration concept to Great Plains tornado count modelling. Filtration ranks were computed efficiently for varieties up to dimension 4, matching theoretical predictions. The tornado application revealed a 29.9% increase in total events over five decades, concentrated in weak categories, with violent events declining. The hierarchical filtration perspective provides a natural framework for decomposing count data by intensity, complementing standard regression approaches in severe weather climatology.

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References

1. Coutinho SC. A primer of algebraic D-modules. Cambridge: Cambridge University Press; 1995. p. 1-207.
2. Björk JE. Rings of differential operators. Amsterdam: North-Holland; 1979. p. 1-374.
3. Kashiwara M. D-modules and microlocal calculus. Translations of Mathematical Monographs. Vol. 217. Providence (RI): American Mathematical Society; 2003. p. 1-254.

4. Brooks HE, Doswell CA III, Kay MP. Climatological estimates of local daily tornado probability for the United States. *Weather Forecast.* 2003;18(4):626-640.
5. Verbout SM, Brooks HE, Leslie LM. Evolving climatology of US tornado outbreaks. *Weather Forecast.* 2006;21(5):701-714.
6. Elsner JB, Jagger TH, Elsner IJ. Tornado intensity estimated from damage path dimensions. *PLoS One.* 2014;9(9):e107571.
7. Tippet MK, Lepore C, Cohen JE. More tornadoes in the most extreme US tornado outbreaks. *Science.* 2016;354(6318):1419-1423.
8. Diffenbaugh NS, Scherer M, Trapp RJ. Robust increases in severe thunderstorm environments in response to greenhouse forcing. *Proc Natl Acad Sci U S A.* 2013;110(41):16361-16366.
9. Bernstein IN. Algebraic theory of D-modules. Moscow: Unpublished Notes; 1971. p. 1-40.
10. Borel A, editor. Algebraic D-modules. Boston: Academic Press; 1987. p. 1-355.
11. Moore TW, DeBoer TA. A review and analysis of possible changes to the climatology of tornadoes in the United States. *Prog Phys Geogr.* 2019;43(3):365-390.
12. Saito M. D-modules on analytic spaces. *Publ Res Inst Math Sci.* 1989;25(3):399-432.
13. Ashley WS, Strader SM. Recipe for disaster: How the dynamic ingredients of risk and exposure are changing the tornado disaster landscape. *Bull Am Meteorol Soc.* 2016;97(5):767-786.
14. Maisonobe P, Sabbah C. D-modules cohérents et holonomes. Paris: Hermann; 1993. p. 1-193.
15. Agee E, Childs S. Adjustments to US tornado climatology for unreported and under-reported events. *Nat Hazards.* 2014;71(3):1519-1530.