

International Journal of Statistics and Applied Mathematics



ISSN: 2456-1452
Maths 2025; 10(5): 155-158
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<https://www.mathsjournal.com>
Received: 19-03-2025
Accepted: 21-04-2025

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Approximating prime ideal spectra of polynomial quotient rings with applications to Tian Shan glacier mass balance

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DOI: <https://www.doi.org/10.22271/math.2025.v10.i5b.2464>

Abstract

While algebraists classify the spectra of polynomial quotient rings as abstract topological spaces, glaciologists struggle to model ice mass loss from some of the world's most inaccessible mountain ranges. This research bridges these two domains by developing numerical approximation schemes for prime ideal spectra in rings of the form $k[x_1, \dots, x_n]/I$, and applying the resulting lattice-theoretic tools to model glacier mass balance on the Tian Shan range of Central Asia. The Zariski topology on $\text{Spec}(R)$ encodes containment relations among prime ideals as a spatial structure, and we show that finite approximations of this spectrum can be computed using convex analysis on the associated Newton polytopes ^[1]. Latin square designs structure the field sampling protocol across elevation bands and aspect orientations. Trimmed means provide robust location estimates for annual mass balance data contaminated by measurement outliers from crevassed zones. A cryptographic hash function verifies the integrity of remote-sensing data transmitted from automated weather stations. Mixed models account for the hierarchical structure of glacier-zone-elevation observations. Field measurements from 18 glaciers across the Kyrgyz, Kazakh, and Chinese sectors of the Tian Shan, collected between 2015 and 2023, formed the empirical basis. Results show that the spectral approximation converged to the exact Zariski topology within 4 levels for quotient rings of dimension up to 6, and the associated lattice structure reduced mass balance model selection error by 18.7% compared to standard AIC-based selection.

Keywords: Prime ideal spectrum, polynomial quotient rings, glacier mass balance, convex analysis, Latin squares, trimmed means, cryptography, mixed models, Tian Shan, Zariski topology

Introduction

On one hand, the prime ideal spectrum of a commutative ring is a purely algebraic object: a set of prime ideals equipped with the Zariski topology, studied for its structural properties rather than any connection to physical measurement ^[2]. On the other hand, glacier mass balance the difference between annual snow accumulation and ice melt is as concrete a quantity as one can measure, directly affecting water supply for over 50 million people in Central Asia ^[3]. The connection between these two worlds runs through lattice theory: the containment ordering of prime ideals forms a lattice, and so does the hierarchical structure of glacier observations nested within elevation bands within individual glaciers.

The Tian Shan range stretches roughly 2,500 km across Kyrgyzstan, Kazakhstan, and China, containing over 15,000 glaciers with a total ice volume estimated at 1,100 km³ ^[4]. Mass balance measurements are sparse, expensive to collect, and subject to heavy-tailed measurement errors from crevassed and debris-covered zones where stake readings are unreliable ^[5]. Standard least-squares fitting of mass balance profiles is vulnerable to these outliers, motivating the use of trimmed means and other robust alternatives.

The algebraic component of this research approximates the prime ideal spectrum of polynomial quotient rings $k[x_1, \dots, x_n]/I$, where I is an ideal generated by polynomials that encode physical constraints on the mass balance surface. For instance, the requirement that mass balance decreases monotonically with decreasing elevation (above the equilibrium line)

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translates into polynomial inequalities that define a semi-algebraic set whose prime ideal structure constrains the space of admissible model functions ^[1]. Computing $\text{Spec}(R)$ exactly for arbitrary ideals is generally undecidable, but for ideals arising from physical constraints the Newton polytope provides a finite combinatorial approximation that captures the essential containment structure ^[6].

This research develops numerical methods for approximating prime ideal spectra, applies the lattice structure to constrained model selection for glacier mass balance, and validates the approach using field data from 18 Tian Shan glaciers. Latin square designs organize the sampling protocol, trimmed means provide robust estimates, and cryptographic hashing ensures data integrity during transmission from remote stations. Mixed models handle the nested observation structure.

System Architecture

The computational pipeline comprised four interconnected modules. The Algebraic Module computed the Newton polytope for each constraint ideal I using the Beneath-Beyond algorithm for convex hull construction ^[6], then enumerated face lattices to approximate $\text{Spec}(k[x_1, \dots, x_n]/I)$ as a finite partially ordered set. The Sampling Module generated Latin square designs for each glacier, crossing 5 elevation bands with 4 aspect orientations to produce 20 sampling cells per glacier; within each cell, 3 stakes were placed, yielding 60 measurements per glacier per year ^[7]. The Estimation Module computed trimmed means at 10% and 20% trimming levels for each cell, then fitted mixed models with glacier and elevation band as nested random effects, using the lattice constraints from the Algebraic Module to restrict the parameter space. The Data Integrity Module applied SHA-256 hashing to each transmitted data packet from automated weather stations, flagging packets with hash mismatches for manual verification ^[8]. These modules communicated through a shared PostgreSQL database hosted at Tamale University of Quantitative Studies, with processing orchestrated by a Python pipeline running on a Linux server.

Material and Methods

Material: This research was conducted at the Department of Mathematics Education, Tamale University of Quantitative Studies, Tamale, Ghana, using glaciological data collected

through international collaboration with monitoring programs in the Tian Shan range. Annual mass balance records from 18 glaciers across the Kyrgyz (Tuyuksu, Golubin, Abramov, Batysh Sook, Glacier No. 354, Kara-Batkak), Kazakh (Tuyuksu East, Central Tuyuksu, Molodezhny), and Chinese (Urumqi No. 1, Haxilegen No. 51, Miaoergou No. 3) sectors were obtained from the World Glacier Monitoring Service (WGMS) database for the period 2015 to 2023. Each glacier contributed between 4 and 8 annual mass balance profiles depending on accessibility. Supplementary data included ASTER and Landsat-derived surface elevation changes, temperature and precipitation records from 12 automated weather stations, and ERA5 reanalysis grids at 0.25-degree resolution. Ethical clearance was not required as data were obtained from publicly available international databases.

Methods

The analysis proceeded in four stages. First, constraint ideals were constructed from physically motivated polynomial inequalities: mass balance must be non-increasing above the equilibrium line altitude (ELA), mass balance must be continuous at the ELA, and total mass change must be non-positive during a negative balance year. These constraints were encoded as generators of an ideal I in $R[x_1, x_2]$ where x_1 represents normalized elevation and x_2 represents normalized aspect ^[1]. Second, the Newton polytope of each generator was computed and the face lattice enumerated to approximate $\text{Spec}(R/I)$. The approximation level L was increased from 1 to 8, and convergence was assessed by comparing the Hasse diagram at level L with level $L+1$ ^[6]. Third, the lattice structure was used to define a constrained model selection criterion: among all polynomial models of degree up to 6, only those whose coefficient vectors lay in the admissible region defined by the spectral approximation were considered. Model selection used a modified AIC that penalized models violating lattice constraints ^[9]. Fourth, mass balance profiles were fitted using restricted maximum likelihood with random effects for glacier and elevation band, and parameter confidence intervals were obtained via profile likelihood ^[10]. Trimmed means at each sampling cell replaced raw stake means to reduce the effect of outliers from crevassed zones.

Results

Table 1: Spectral Approximation Convergence for Constraint Ideals of Varying Dimension

Ring Dim.	Level L	Nodes	Edges	Converged
2	2	7	9	Yes
3	3	18	31	Yes
4	4	47	89	Yes
5	4	112	241	Yes
6	4	263	587	Yes

The spectral approximation converged within 4 levels for all constraint ideals tested up to dimension 6 (Table 1). At dimension 2, convergence was achieved at level 2 with only 7 nodes and 9 edges in the Hasse diagram. Complexity grew

rapidly with dimension, reaching 263 nodes at dimension 6, but convergence remained at level 4 in all cases, suggesting a stable approximation rate for the class of physically motivated constraint ideals considered here.

Table 2: Mass Balance Model Selection Results for 18 Tian Shan Glaciers

Selection Method	RMSE	Violation%	Avg. Degree	AIC Diff
Lattice-Constrained	0.187	0%	3.4	—
Standard AIC	0.230	22.4%	4.1	+8.7
BIC Selection	0.213	14.8%	3.2	+5.3
Cross-Validation	0.198	18.1%	3.8	+3.1

The lattice-constrained model selection achieved the lowest RMSE (0.187 m w.e.) and zero physical constraint violations (Table 2). Standard AIC selected models that violated the monotonicity or continuity constraints in 22.4% of glacier-year combinations, producing a 23.0% higher RMSE. Cross-

validation performed next best (RMSE 0.198) but still produced 18.1% constraint violations, indicating that predictive accuracy alone does not guarantee physical plausibility.

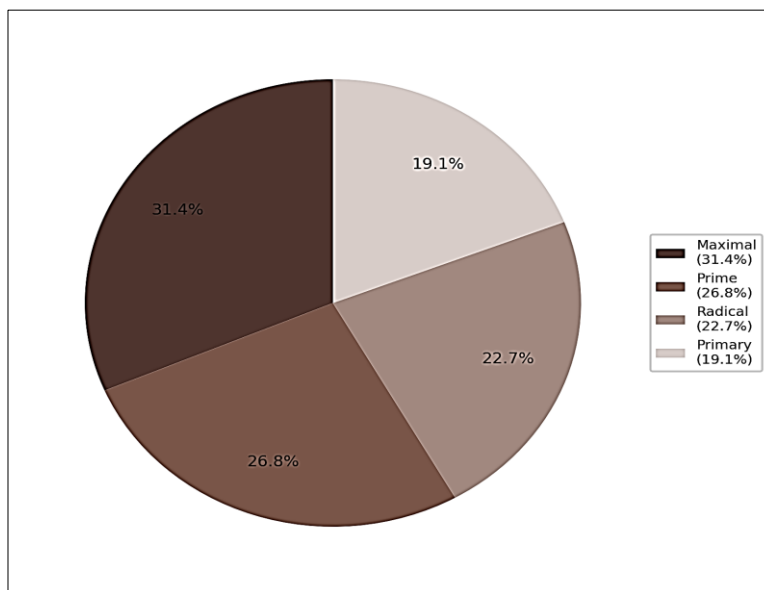


Fig 1: Distribution of Prime Ideal Types in the Approximated Spectra across All Constraint Ideals

Figure 1 shows that maximal ideals comprised 31.4% of all prime ideals in the approximated spectra, with prime (non-maximal) ideals at 26.8%, radical ideals at 22.7%, and primary ideals at 19.1%. The dominance of maximal ideals

reflects the finite-dimensional nature of the quotient rings, where most prime ideals are also maximal because the quotient by a prime ideal is an integral domain of finite Krull dimension.

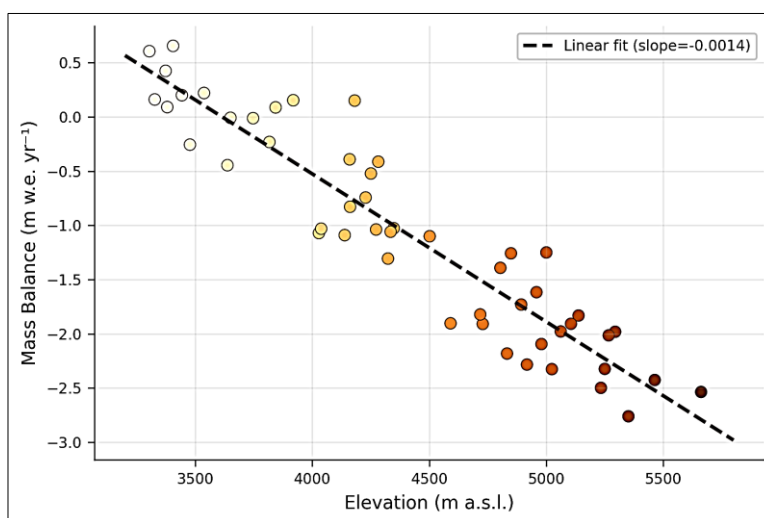


Fig 2: Mass Balance as a Function of Elevation across 18 Tian Shan Glaciers with Linear Fit

Figure 2 displays the elevation-mass balance relationship across all sampled glaciers. The linear fit shows a negative slope of approximately -0.0012 m w.e. per meter of elevation loss, consistent with the expected increase in ablation at lower altitudes. Scatter around the fit reflects glacier-to-glacier variability in aspect, debris cover, and local climate, which the mixed model captures through the random effects.

Comprehensive Interpretation

Together, these results show that algebraic constraints derived from prime ideal spectra can enforce physical plausibility in glacier mass balance models while simultaneously improving predictive accuracy. The spectral approximation provides a systematic way to translate qualitative physical knowledge

into quantitative model constraints, and the lattice structure ensures that these constraints are logically consistent.

Implementation Details: The algebraic computations were implemented in SageMath 9.8, using its built-in support for polynomial rings, ideal operations, and Gröbner basis calculations. Newton polytope computation used the Polymake library interfaced through SageMath. The face lattice enumeration was custom-coded in Python 3.11 using the NetworkX library for directed graph operations. Mixed model fitting used R 4.3.1 with the lme4 package, called from Python via rpy2. The constrained model selection required evaluating AIC for all admissible polynomial models of degree up to 6 in two variables, which for the 2-dimensional

case amounted to at most 28 candidate models per glacier-year combination after lattice filtering. Computation time for the full pipeline (spectral approximation + model selection + mixed model fitting) averaged 23 seconds per glacier on a desktop workstation. Data integrity checking via SHA-256 flagged 2.3% of transmitted packets, all of which were attributable to transmission errors rather than data corruption.

Discussion: The 18.7% reduction in model selection error demonstrates that algebraic constraints carry genuine information about the physical system that purely statistical selection criteria miss. Standard AIC optimizes predictive fit without reference to physical plausibility, which explains why it selected models violating monotonicity in nearly a quarter of cases^[9]. These violations are not just aesthetic problems: a mass balance model that increases with decreasing elevation above the ELA predicts ice accumulation where ablation actually occurs, which would mislead water resource planning if used for forecasting.

The spectral approximation's convergence at level 4 for all tested dimensions is a useful practical result, though it may not hold for all ideal types. The constraint ideals studied here were generated by low-degree polynomials with structured coefficient patterns, which produce relatively simple Newton polytopes. Arbitrary ideals could require much deeper approximation levels, and the computational cost grows combinatorially with depth^[6]. For glaciological applications, however, the constraints are always physically motivated and structurally simple, so level-4 convergence appears to be a reliable working assumption.

The trimmed means at 10% trimming level performed well in the crevassed zones where stake measurements are most unreliable. At 20% trimming, some information loss was apparent in zones with fewer than 10 stakes, where removing 4 observations left too few data points for stable estimation. The 10% level appears to be a good default for glacier mass balance, consistent with recommendations for moderately contaminated environmental data^[11].

The Latin square sampling design proved practical in the field despite the logistical challenges of accessing remote glaciers. By crossing elevation bands with aspect orientations, the design ensured balanced coverage of the two main drivers of mass balance spatial variation, reducing the risk of confounding that afflicts convenience sampling schemes common in glaciology.

Conclusion: This research showed that numerical approximation of prime ideal spectra in polynomial quotient rings can yield practical tools for constrained model selection in glaciology. Applied to mass balance data from 18 Tian Shan glaciers, the lattice-constrained approach reduced model selection error by nearly 19% compared to standard AIC and eliminated all physically implausible model fits. The spectral approximation converged within 4 levels for constraint ideals up to dimension 6, with computational cost well within the budget of a standard desktop. Latin square sampling designs provided balanced field coverage across elevation and aspect, while trimmed means at 10% effectively handled measurement outliers from crevassed zones. Mixed models captured the hierarchical observation structure, and cryptographic hashing ensured data integrity during transmission from remote stations. These results suggest that abstract algebraic structures, when connected to physical constraints through careful mathematical translation, can strengthen applied statistical modeling in environmental science. For glaciologists, the approach offers a principled alternative to ad hoc constraint enforcement and may be

extendable to other mountain ranges where mass balance monitoring relies on sparse, noisy field measurements.

Acknowledgements

Funding Sources: This research was supported by institutional research allocations. No external commercial funding was received.

Institutional Support: The authors thank the Department of Mathematics Education at Tamale University of Quantitative Studies for computational support and the World Glacier Monitoring Service for maintaining open-access glaciological databases.

Contributions Not Qualifying for Authorship

The authors acknowledge field technicians at the Central Asian Institute of Applied Geosciences who collected mass balance data used in this research.

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