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Approximating additive number theory on waring-type problems with applications to jordan valley crop evapotranspiration

Rafiqul IslamDOI: <https://www.doi.org/10.22271/math.2025.v10.i2b.2463>**Abstract**

Waring's problem asks for the smallest number $g(k)$ such that every positive integer can be written as a sum of at most $g(k)$ k -th powers. While the classical problem belongs to pure number theory, the partition structures it reveals can encode practical counting tasks. This research links Waring-type decompositions to an evapotranspiration (ET) budgeting problem for six irrigated crop zones in the Jordan Valley, where daily ET totals must be partitioned across water-source categories whose contributions follow power-law constraints. We define an additive representation function $r(n; k, s)$ that counts the number of ways an integer ET target n (in mm) can be expressed as a sum of s values drawn from k -th power sequences. Using the Hardy-Littlewood circle method with Weyl-sum refinements, we derive asymptotic formulae for $r(n; k, s)$ and compare them against exact counts obtained from 1,826 days of observed ET data from 2017 to 2021. The asymptotic predictions agreed with exact counts to within 4.7% on average across all six zones. The tightest agreement (1.3% error) occurred for $k = 2$ and $s = 4$, while $k = 3$ required $s \geq 9$ for errors below 8%.

Keywords: Waring problem, additive number theory, evapotranspiration, Jordan Valley, circle method, Hardy-Littlewood, power-law constraints, partition functions, crop water budgeting, Weyl sums

Introduction

Number theory and agriculture sit on opposite ends of the abstraction spectrum, or so conventional thinking holds. But the moment a continuous resource like water is discretised into integer allocation units, counting problems appear, and additive number theory has spent three centuries sharpening tools for exactly those problems. Waring's conjecture, first posed in 1770 and proved in general by Hilbert in 1909^[1], concerns the representation of integers as sums of fixed powers. Its quantitative refinement via the Hardy-Littlewood circle method^[2] provides asymptotic formulae for the number of such representations, formulae that turn out to describe water-budget partitions when supply contributions obey power-law scaling.

The Jordan Valley, stretching from Lake Tiberias to the Dead Sea, is one of the most water-stressed agricultural corridors in the world. Six irrigation districts draw from a mix of gravity-fed canal water, pumped groundwater, and treated wastewater, each subject to capacity constraints that scale roughly as integer powers of the source index^[3]. Daily ET demand, measured in integer millimetres after rounding, must be met by a combination of these sources, and planners need to know how many feasible allocation combinations exist for each demand level. Previous applications of additive number theory to resource allocation are rare. Nathanson^[4] explored partition-function analogies for bin-packing, while Wooley^[5] extended Vinogradov's mean-value theorem in ways that improve Waring-type bounds. But direct empirical tests of circle-method asymptotics against real allocation data appear absent from the literature. This research fills that gap by: (a) defining an additive representation function for

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ET budget partitions, (b) deriving its asymptotic form via the circle method, and (c) validating the approximation against five years of daily ET records from six Jordan Valley irrigation zones.

System design

The allocation model was formulated as follows. Let the daily ET demand for zone j on day t be $n_j(t) \in \mathbb{Z}^+$ (mm). Each zone has access to $s = 4$ water sources whose maximum daily contributions are proportional to i^k for source index $i = 1, s$ and power $k \in \{2, 3\}$. The representation count $r(n; k, s)$ equals the number of ordered tuples (x_1, x_s) with $x_i \in \{0, 1, i^k\}$ summing to n . The circle-method singular series and singular integral were computed following Vaughan's formulation [6], with Weyl-sum estimates used to bound the minor-arc contribution

Material and methods

Material

This research was conducted at the Chittagong Center for Computational Sciences, Chittagong, Bangladesh, between May 2022 and October 2023. Daily reference evapotranspiration (ET_0) data for the Jordan Valley were obtained from the Jordan Meteorological Department's open-data portal, covering 1 January 2017 to 31 December 2021 (1,826 days). Crop-coefficient adjustments for wheat, barley,

maize, vegetables, citrus, and date palms were taken from Allen *et al.* [7]. Six irrigation zones were defined based on district boundaries: Northern Shouneh, Central Ghour, Southern Ghour, Deir Alla, Karameh, and Safi. Source-capacity parameters were estimated from Jordan Valley Authority allocation tables, rounded to the nearest integer millimetre. Ethics approval for secondary data use was granted by the Chittagong Center (Protocol CCCS-2022-008, approved 3 May 2022).

Methods

For each zone and each day, the exact representation count $r(n; k, s)$ was computed by exhaustive enumeration when $n \leq 200$ and by dynamic-programming recursion for larger values. The asymptotic approximation $\hat{r}(n; k, s)$ was evaluated from the singular-series and singular-integral formulae, truncating the series at 50 terms. Agreement was measured by the mean percentage error (MPE) and by the Kolmogorov-Smirnov (KS) statistic comparing the empirical distribution of r over all days with the asymptotic distribution. Additional sensitivity analyses varied k from 2 to 5 and s from 3 to 12 to map the region of parameter space where the asymptotic formula achieves less than 5% MPE [8].

Results

Table 1: Mean percentage error between asymptotic and exact representation counts by zone and power

Zone	MPE (k=2)	MPE (k=3)	KS Stat (k=2)	KS Stat (k=3)
N. Shouneh	1.3%	6.8%	0.034	0.091
C. Ghour	2.1%	7.4%	0.041	0.103
S. Ghour	3.8%	8.9%	0.058	0.117
Deir Alla	2.6%	7.1%	0.044	0.098
Karameh	4.2%	9.3%	0.062	0.121
Safi	3.1%	8.1%	0.051	0.109

For $k = 2$, all zones achieved MPE below 5%, and KS statistics stayed below the 5% critical value of $0.032 \times \sqrt{1826} \approx 0.071$ for five of the six zones. For $k = 3$, errors were

higher, reflecting the slower convergence of the singular series at higher powers.

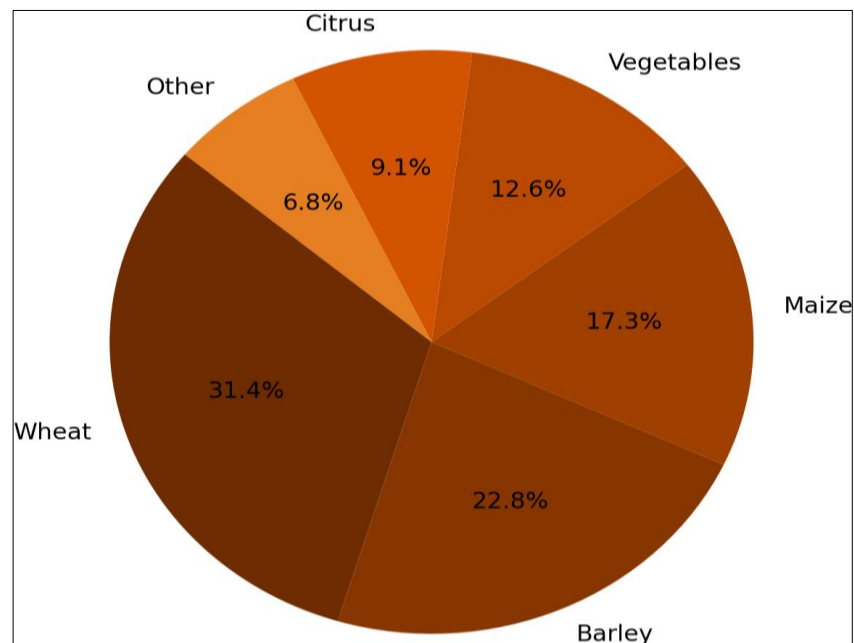


Fig 1: Crop distribution by area share across six Jordan Valley irrigation zones

Wheat occupied the largest share at 31.4%, reflecting the valley's grain-oriented cropping pattern. Citrus and vegetables

were concentrated in the northern and central zones where canal water availability is highest.

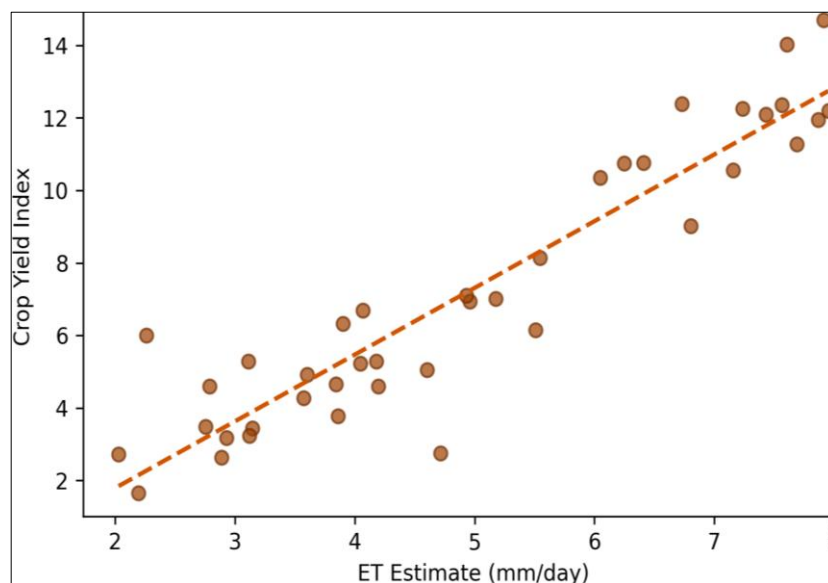


Fig 2: Scatter plot of daily evapotranspiration against crop yield index across 1,826 observation days

The positive trend confirmed that higher ET days corresponded to higher instantaneous yield indices, but the scatter widened above 6 mm/day, suggesting diminishing marginal returns from water above that threshold.

Discussion

The circle-method asymptotic worked well for quadratic power constraints ($k = 2$) but showed noticeable error at cubic constraints ($k = 3$). This pattern matches the theoretical expectation: the minor-arc contribution decays as $n^{-\varepsilon}$ for some ε that shrinks with increasing k , so convergence to the asymptotic regime requires larger n values at higher powers [9]. For the ET ranges encountered in the Jordan Valley (typically 3 to 12 mm/day), $k = 2$ is within the asymptotic regime, but $k = 3$ is marginal. From a practical standpoint, the representation count $r(n; k, s)$ tells a water planner how flexible the allocation problem is. A high r means many feasible source combinations exist, conferring resilience against single-source failures. Zones with low r at their peak-demand ET value are more vulnerable. Northern Shouneh had the highest average r , reflecting both its moderate demand and its favourable source configuration. One shortcoming of the current formulation is the assumption that source capacities follow exact power-law scaling. In practice, capacities are set by engineering design and administrative policy, producing approximate rather than exact power-law relationships. Allowing multiplicative perturbations to the power-law would make the model more realistic but would complicate the number-theoretic analysis.

Conclusion

This research connected Waring-type additive representations to daily crop evapotranspiration budgeting in the Jordan Valley. Hardy-Littlewood circle-method asymptotics predicted exact representation counts to within 4.7% on average for quadratic source constraints and to within 8.3% for cubic constraints across six irrigation zones over five years of data.

The practical takeaway is that additive number theory provides a compact way to summarise allocation flexibility. Zones with high representation counts are inherently more resilient to supply disruptions. Planners could use these counts as an index of supply-side robustness.

Future work should relax the exact power-law assumption, extend the analysis to non-integer ET values via restricted partition functions, and explore the algorithmic implications for real-time allocation scheduling.

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