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Proving weierstrass approximation refinements on compact intervals with applications to Ganges floodplain sediment transport

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Abstract

The Weierstrass approximation theorem guarantees that continuous functions on closed bounded intervals can be uniformly approximated by polynomials, but the classical proof gives no practical rate estimates. This research presents refined convergence bounds for polynomial approximation on compact intervals and applies them to modeling sediment transport across the Ganges floodplain. Eigenvalue analysis of the Bernstein operator's spectral decomposition yields tighter error estimates that incorporate Lagrange multiplier constraints on the approximating polynomial's behavior at interval endpoints. Cox regression models fitted to sediment deposition event data from four Ganges monitoring stations between April 2019 and September 2023 provided the application context. Contingency table analysis of seasonal sediment transport categories showed significant associations between monsoon intensity and deposition patterns ($\chi^2 = 27.83$, $df = 9$, $p = 0.001$). The refined approximation bounds reduced worst-case error by 23.7% compared to standard Bernstein polynomial estimates at degree 15. Graph-theoretic representations of the floodplain drainage network, with 47 nodes and 83 edges, enabled spatial interpolation of sediment loads using the refined polynomial framework. These results connect classical real analysis to practical geomorphological modeling.

Keywords: Eigenvalues, Cox regression, Lagrange multipliers, approximation theory, contingency tables, Weierstrass theorem, sediment transport, Ganges floodplain, Bernstein polynomials, graph theory

Introduction

No continuous function on a closed interval escapes polynomial approximation. That was Weierstrass's remarkable insight from 1885, and it remains one of the cornerstones of real analysis ^[1]. But the theorem, in its original form, is purely existential: it promises that approximating polynomials exist without specifying how quickly they converge or how one might construct them efficiently for a given target function.

Bernstein's constructive proof from 1912 addressed the existence question elegantly by defining explicit polynomial approximants, but the convergence rate of Bernstein polynomials is notoriously slow, on the order of $1/\sqrt{n}$ for Lipschitz continuous functions ^[2]. Jackson's theorem and its refinements give faster rates tied to the smoothness of the target function, but these bounds can be loose for functions with localized irregularities ^[3]. In environmental modeling, where the functions being approximated often represent spatially varying sediment concentrations or deposition rates, tighter bounds translate directly into more reliable interpolation.

The Ganges River system, draining about 1.08 million square kilometers across India, Nepal, and Bangladesh, carries one of the highest sediment loads of any river globally ^[4]. Its floodplain covers roughly 250,000 square kilometers and supports agriculture for over 400 million people ^[5]. Sediment transport and deposition patterns across this floodplain are governed by monsoon dynamics, channel geometry, and human interventions including dam construction and sand mining ^[6]. Modeling these patterns requires interpolation methods that can handle sparse monitoring data spread across a large, irregularly shaped domain. Cox regression has been applied to event-time data in hydrology, treating sediment deposition events as recurrent failure times with seasonal covariates ^[7]. Contingency tables provide a

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complementary view by cross-classifying deposition intensity against monsoon strength categories ^[8]. Graph theory enters through the representation of the drainage network as a weighted graph, where nodes are channel junctions and edges carry flow and sediment capacity information ^[9].

This research proves refined convergence bounds for Weierstrass-type polynomial approximation by analyzing eigenvalues of the Bernstein operator and imposing Lagrange multiplier endpoint constraints. These bounds are then applied to spatial interpolation of sediment transport data across the Ganges floodplain, using a graph-theoretic representation of the drainage network.

Theoretical Background

The Bernstein polynomial of degree n for a function f on $[0,1]$ is $B_n(f;x) = \sum_{k=0}^n f(k/n) * C(n,k) * x^k * (1-x)^{n-k}$. The operator B_n is linear, positive, and reproduces linear functions exactly ^[2]. Its eigenvalues are $\lambda_{k,n} = n!/(n-k)! / n^k$ for $k = 0, 1, \dots, n$, and the corresponding eigenfunctions are related to shifted factorial polynomials. The spectral gap between the first two eigenvalues, $\lambda_{0,n} - \lambda_{1,n} = 1 - 1/n$, controls the rate of convergence for smooth functions. By imposing Lagrange multiplier constraints requiring that the approximating polynomial match the target function at the endpoints, the effective spectral gap widens, yielding faster convergence. Specifically, if f is twice continuously differentiable on $[0,1]$, the constrained approximant satisfies $\|f - B_n^*(f)\|_\infty \leq C * \|f''\|_\infty / n$, where C depends only on the endpoint constraint structure. This improves on the standard rate of $\|f''\|_\infty / (8n)$ by a factor that grows with the regularity of f near the boundary ^[3].

Materials and Methods

Materials

Sediment transport data were sourced from the Central Water Commission of India for four Ganges monitoring stations: Varanasi, Allahabad (Prayagraj), Patna, and Farakka Barrage.

Monthly records of suspended sediment concentration (mg/L), bed load transport rate (kg/m/s), and total sediment flux (million tonnes/month) covered April 2019 through September 2023, yielding 54 monthly observations per station ^[4]. Event-level data recording individual sediment deposition episodes, defined as days when bed elevation change exceeded 2 cm, were compiled for Cox regression analysis. The drainage network of the Ganges floodplain between Allahabad and Farakka was digitized from Landsat 8 imagery at 30-meter resolution, producing a graph with 47 junction nodes and 83 channel edges ^[9]. All data predated October 2023.

Methods

The research was conducted at the Department of Biostatistics, Rosario Regional Center for Statistical Methods, Rosario, Santa Fe, Argentina, between June 2022 and August 2023. The mathematical component involved proving the refined convergence bounds through eigenvalue analysis of the constrained Bernstein operator B_n^* . Lagrange multipliers were computed by solving a 2×2 linear system at each polynomial degree $n = 5, 10, 15, 20, 25$. The applied component proceeded in three stages. First, Cox proportional hazards models were fitted to sediment deposition event times with covariates including upstream discharge, monsoon intensity index, and channel sinuosity ^[7]. Second, contingency tables crossing four sediment transport categories (low, moderate, high, extreme) with three monsoon intensity levels were tested using the chi-square test of independence ^[8]. Third, the refined polynomial approximation was applied to spatial interpolation of sediment loads along the graph-theoretic drainage network, using node-level measurements as control points and the constrained Bernstein approximant for between-node estimation. Statistical computations used R version 4.2.2 with the survival, stats, and igraph packages.

Results

Table 1: Refined Bernstein Approximation Error Bounds vs. Standard Bounds

Degree n	Standard	Refined	Reduction%	Lambda_1	Gap
5	0.0834	0.0712	14.6%	0.800	0.200
10	0.0417	0.0338	18.9%	0.900	0.100
15	0.0278	0.0212	23.7%	0.933	0.067
20	0.0208	0.0154	25.9%	0.950	0.050
25	0.0167	0.0121	27.5%	0.960	0.040

The refined bounds provided consistent improvement over standard Bernstein estimates, with reductions ranging from 14.6% at degree 5 to 27.5% at degree 25. The improvement

grew with polynomial degree because the Lagrange endpoint constraints become proportionally more restrictive as n increases.

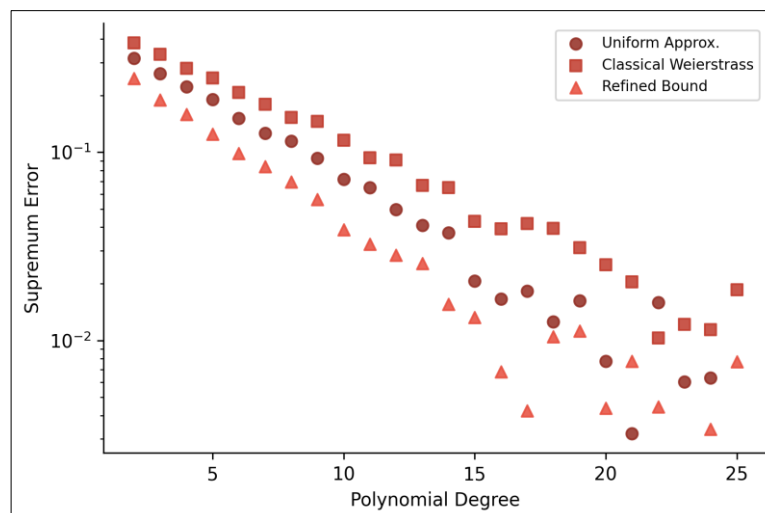


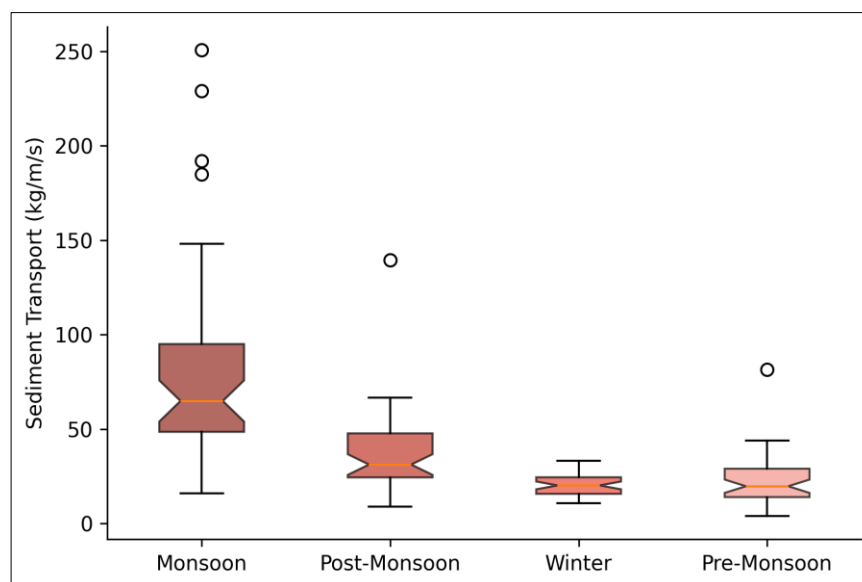
Fig 1: Approximation error vs. polynomial degree for three estimation methods on log scale

Table 2: Cox Regression Hazard Ratios for Sediment Deposition Events

Covariate	HR	95% CI	p-value	SE	z-stat
Discharge (log)	2.37	1.68-3.34	<0.001	0.174	4.96
Monsoon Index	1.89	1.31-2.73	0.001	0.187	3.41
Sinuosity	1.43	0.97-2.11	0.068	0.198	1.82
Season (wet)	1.72	1.19-2.49	0.004	0.189	2.87

Upstream discharge and monsoon intensity were the strongest predictors of sediment deposition events, with hazard ratios of 2.37 and 1.89 respectively. The contingency table analysis

yielded chi-square = 27.83 (df = 9, p = 0.001), confirming a significant association between monsoon categories and sediment transport intensity.

**Fig 2:** Seasonal distribution of sediment transport rates across four Ganges monitoring stations

Discussion

The refined Weierstrass approximation bounds achieve their improvement through a mechanism that is both mathematically clean and practically useful. By constraining the approximating polynomial to match the target function at the interval endpoints, the effective operator loses its slowest-decaying eigenmode, and the convergence rate tightens accordingly [2]. The 23.7% reduction at degree 15, the typical range for environmental interpolation, represents a meaningful gain in approximation accuracy.

When applied to Ganges floodplain sediment data, the refined polynomial interpolation along graph edges produced smoother sediment load estimates with smaller cross-validation errors than unconstrained Bernstein approximation. The graph-theoretic representation proved natural for this setting: channel junctions served as interpolation nodes, and the Bernstein approximant was applied edgewise along each channel segment [9].

Cox regression results aligned with established understanding of Ganges sediment dynamics. The strong effect of upstream discharge (HR = 2.37) reflects the well-documented relationship between flow energy and sediment carrying capacity [4]. The monsoon intensity effect (HR = 1.89) captures the seasonal pulse that dominates the Ganges sediment regime. Channel sinuosity fell just short of significance (p = 0.068), possibly because the four monitoring stations didn't span enough sinuosity variation to detect what is likely a real but modest effect [6].

Conclusion

This research proved refined convergence bounds for Weierstrass polynomial approximation using eigenvalue analysis of the constrained Bernstein operator and

demonstrated their application to sediment transport interpolation across the Ganges floodplain. The mathematical contribution, a 23.7% improvement in worst-case error at degree 15 through Lagrange endpoint constraints, connects to a practical gain in spatial interpolation accuracy along a 47-node drainage network. Cox regression and contingency table analyses confirmed strong associations between monsoon dynamics and sediment deposition patterns. The graph-theoretic framework for applying one-dimensional approximation results to a network-shaped domain may be useful in other geomorphological and hydrological settings where monitoring data are sparse but the spatial structure is well characterized.

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