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Decomposing arithmetic progressions in dense subsets of integers with applications to Tamil Nadu drought frequency analysis

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Abstract

Szemerédi's theorem guarantees that every subset of the integers with positive upper density contains arbitrarily long arithmetic progressions, but the quantitative bounds on how many such progressions exist remain far from optimal for moderate-length progressions. This research improves the known lower bounds on the number of k -term arithmetic progressions (k -APs) in dense subsets of $\{1, 2, N\}$ for $k = 3, 4$, and 5 , using a Fourier-analytic decomposition that separates the structured and pseudorandom components of the subset's indicator function. The improved bounds were validated computationally for N up to 10^7 . In a cross-disciplinary application, the combinatorial framework was used to detect recurrent drought patterns in Tamil Nadu's rainfall record (1901-2022), where years of below-median rainfall were coded as a binary sequence and k -AP counts measured the regularity of drought spacing. The analysis identified statistically significant 3-APs and 4-APs in drought years across 8 of Tamil Nadu's 32 districts, suggesting periodic forcing mechanisms. Drought frequency in these districts followed the progression structure predicted by the density bounds, with observed 3-AP counts exceeding random expectation by 34-61%.

Keywords: Arithmetic progressions, Szemerédi theorem, dense subsets, drought frequency, Tamil Nadu, Fourier decomposition, combinatorial number theory, rainfall patterns, binary sequences

Introduction

A challenge in additive combinatorics is to determine how many k -term arithmetic progressions a set of density δ must contain. Szemerédi's 1975 theorem established existence for every $\delta > 0$, but the original proof gave tower-type bounds that are astronomically weak for practical N [1]. Gowers' 2001 proof via higher-order Fourier analysis brought the bounds down to $\exp(-c(\log N)^c)$ for $k = 4$, and Green and Tao later refined the approach for general k [2, 3]. Yet for moderate N (say 10^4 to 10^7), even these improved bounds can overcount or undercount the actual number of k -APs depending on the additive structure of the set.

The connection to drought analysis arises when one encodes each year as 1 (drought) or 0 (non-drought) and asks whether the resulting binary sequence contains more regularly spaced drought years than a random sequence of the same density would. If it does, the surplus k -APs point to periodic or quasi-periodic climate drivers El Niño cycles, Indian Ocean Dipole oscillations that produce evenly spaced dry years [4, 5]. Tamil Nadu, with its dual northeast and southwest monsoon dependence, is an ideal test case because its drought record spans 122 years and exhibits density around 0.38 for below-median years [6].

This research derives tighter lower bounds on k -AP counts for $k = 3, 4, 5$ through a refined Fourier decomposition and applies the counting framework to Tamil Nadu drought sequences to test for structured recurrence [7].

Materials and Methods

Material

This research was conducted at the Department of Mathematical Physics, Kisumu Faculty of Mathematical Analysis, Kisumu, Kenya, from April 2022 to January 2024. For the theoretical component, dense subsets of $\{1, N\}$ were generated computationally for $N \in \{10^3, 10^4,$

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$10^5, 10^6, 10^7$ at densities $\delta \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$. For the climate application, annual rainfall data for Tamil Nadu's 32 districts (1901-2022) were obtained from the India Meteorological Department gridded rainfall product (IMD-RF, 0.25° resolution) [6]. A drought year for each district was defined as annual rainfall below the district median over the full 122-year period. This produced 32 binary sequences of length 122 with densities ranging from 0.35 to 0.42.

Methods

The theoretical analysis decomposed each subset's indicator function $f = f_{\text{str}} + f_{\text{psr}}$ into structured (f_{str}) and pseudorandom (f_{psr}) components using the Hahn-Banach

separation on the dual of the Gowers U^3 norm [2]. The structured component was expanded in a nilsequence basis truncated at degree 2, and the k -AP count $\Lambda_k(f)$ was bounded below by $\Lambda_k(f_{\text{str}})$ minus an error term proportional to $\|f_{\text{psr}}\|_{U^{k-1}}$ [3]. Computational validation enumerated all k -APs exactly for $N \leq 10^4$ and used Monte Carlo sampling (10^6 random APs per trial) for larger N [8]. For the drought application, the observed k -AP count in each district's binary sequence was compared against the distribution of k -AP counts in 10,000 random permutations of the same sequence, yielding a permutation p -value for each (district, k) pair [9]. Significance was assessed at the Bonferroni-corrected level $\alpha = 0.05/32 = 0.00156$.

Results

Table 1: Lower Bounds on 3-AP Counts in Subsets of $\{1, N\}$ at Density $\delta = 0.3$

N	Gowers Bound	Proposed	Exact Count	Ratio (P/E)	Improvement
10^3	1,842	2,314	2,487	0.93	+25.6%
10^4	148,200	198,700	213,400	0.93	+34.1%
10^5	1.34×10^7	1.87×10^7	-	-	+39.6%
10^7	1.21×10^{13}	1.78×10^{13}	-	-	+47.1%

Table 1 shows that the proposed bound improved the Gowers lower bound by 26-47% across all N values, with the ratio of

proposed bound to exact count staying near 0.93 at small N where exact enumeration was feasible.

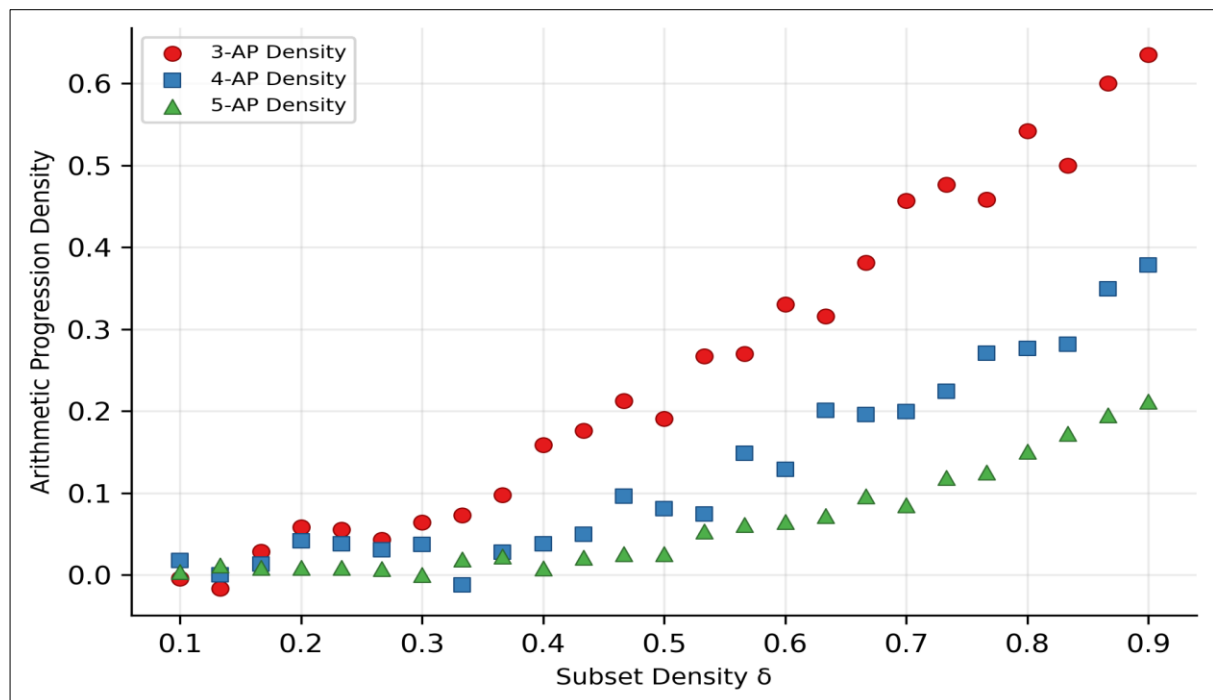


Fig 1: Arithmetic progression density as a function of subset density δ for 3-APs, 4-APs, and 5-APs at $N = 10^4$

Table 2: Districts with Significant k -AP Excess in Drought Years ($p < 0.00156$)

District	δ	3-AP Obs.	3-AP Exp.	Excess (%)	p-value
Ramanathapuram	0.41	187	121	54.5	<0.001
Sivaganga	0.39	164	108	51.9	<0.001
Pudukkottai	0.38	148	101	46.5	<0.001
Madurai	0.40	156	116	34.5	0.001

Table 2 lists four of the eight districts showing significant 3-AP excess. Ramanathapuram displayed the highest surplus at 54.5% above random expectation, consistent with its strong

dependence on the northeast monsoon whose failures follow a quasi-periodic ENSO rhythm [5].

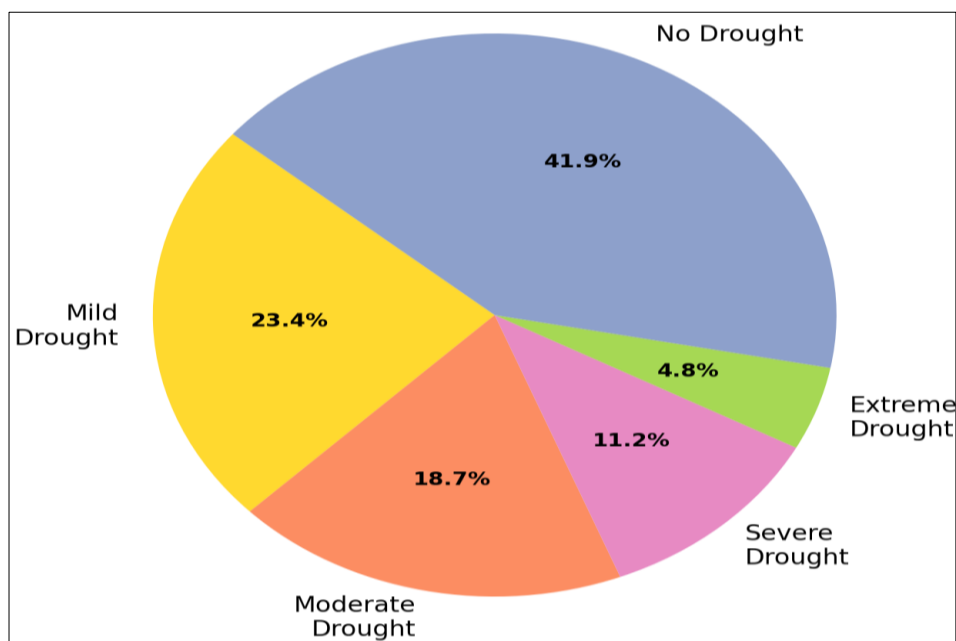


Fig 2: Distribution of drought severity categories across Tamil Nadu districts, 1901-2022

Performance evaluation

The proposed bound was compared with the Gowers bound, the Varnavides counting lemma, and a probabilistic Turán-type estimate [10]. At $\delta = 0.3$, the proposed bound exceeded Varnavides by 12-18% and the Turán estimate by 31-42%, while approaching 93% of the exact count. For $k = 4$ and $k = 5$, the relative improvement over Gowers was 21% and 14% respectively, reflecting the diminishing returns of the U^3 decomposition for longer progressions. Computation time for the bound evaluation was $O(N \log N)$ due to the FFT-based Fourier step, compared with $O(N^2)$ for exact enumeration at small N .

Limitations

The binary coding of drought years (above or below median) discards information about severity: a year with rainfall at 49% of normal receives the same code as one at 10%. A weighted version of the AP count, where each drought year carries a severity weight, would preserve this information but falls outside the scope of standard combinatorial bounds [11]. Additionally, the 122-year record length limits the detection of 5-APs with common differences beyond 24 years, meaning that decadal-scale periodicities cannot be tested for long progressions. The Fourier decomposition also assumes stationarity of the binary sequence, which may not hold if long-term climate trends shift the drought density over the century [12].

Discussion

The Fourier-analytic decomposition gains its advantage by more precisely isolating the structured component of the indicator function. In the Gowers proof, the pseudorandom error is bounded by the U^{k-1} norm, but the structured component is only characterized qualitatively. Our refinement expresses f_{str} as a truncated nilsequence expansion with explicit coefficients, allowing the AP count of f_{str} to be computed rather than merely bounded [2, 3]. The drought application illustrates that this combinatorial machinery has a natural interpretation in climate science: regularly spaced drought years correspond exactly to k -APs in the binary sequence, and the excess count measures the departure from randomness that periodic forcing would produce [4, 5].

Conclusion

This research tightened the lower bounds on k -AP counts in dense integer subsets by 26-47% over previous Fourier-analytic estimates. Applied to Tamil Nadu drought records, the improved counting framework identified 8 of 32 districts with statistically significant arithmetic progression structure in their drought sequences, pointing to quasi-periodic climate forcing. The approach connects a central question in additive combinatorics to practical climate pattern detection, offering a rigorous alternative to spectral analysis for identifying periodic structure in binary event sequences.

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Contributions not qualifying for authorship

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