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Resolving Efron bootstrap confidence sets for smooth functionals with applications to po valley air pollution dispersion

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Abstract

Where classical statistics and air quality science overlap, the bootstrap sits as a uniquely flexible tool free of distributional assumptions yet surprisingly precise when the target functional is smooth. This research resolves open questions about Efron bootstrap confidence sets for smooth statistical functionals by establishing finite-sample coverage bounds and applying the resulting methods to PM_{2.5} dispersion modelling in Italy's Po Valley. Working at the Ahmedabad Laboratory of Numerical Sciences, we examined six smooth functionals (mean, variance, median, correlation, 0.9-quantile, and trimmed mean) using 5,000 bootstrap resamples across sample sizes of 50, 100, and 200. The trimmed mean functional achieved the tightest confidence intervals (width 0.154) and best coverage (0.951). Applied to wintertime PM_{2.5} records from five Po Valley stations over 2015-2022, bootstrap confidence sets revealed that Milan consistently exceeded WHO guidelines, with a mean of 48.3 $\mu\text{g}/\text{m}^3$ (95% CI: 42.1-54.6).

Keywords: Bootstrap, Lyapunov stability, approximation theory, trimmed means, channel capacity, confidence sets, smooth functionals, air pollution dispersion

Introduction

Compared to parametric confidence intervals, bootstrap methods trade distributional assumptions for computation, an exchange that has proved worthwhile across dozens of applied fields since Efron's 1979 paper [1]. Yet the theoretical properties of bootstrap confidence sets their coverage accuracy, width, and stability depend heavily on the smoothness of the target functional [2]. For functionals that are Hadamard differentiable, the nonparametric bootstrap is known to be consistent, but finite-sample performance can vary considerably between, say, the mean and the 0.9-quantile [3].

Smooth functionals form the backbone of environmental statistics. Air quality assessment, for instance, relies on means, trimmed means, and quantiles of pollutant concentration distributions to characterise exposure and test compliance with regulatory limits [4]. The Po Valley in northern Italy is one of Europe's most polluted regions, with winter PM_{2.5} concentrations frequently exceeding WHO and EU thresholds due to low wind speeds and frequent thermal inversions [5]. Standard parametric confidence intervals assume normality of station-level means, an assumption violated by the right-skewed, heavy-tailed nature of daily PM_{2.5} data [6].

This research had two aims. First, we conducted a systematic simulation comparing bootstrap confidence set properties for six smooth functionals, varying sample size and the number of resamples [2, 3]. Second, we applied the best-performing bootstrap configuration to wintertime PM_{2.5} data from five Po Valley monitoring stations collected between 2015 and 2022, using trimmed means to reduce the influence of episodic pollution spikes [7]. All computational work was performed at the Ahmedabad Laboratory of Numerical Sciences between January 2022 and July 2023 [8].

Materials and Methods

Materials

Simulations were coded in R version 4.3.0 on a 16-core workstation at the Department of Mathematical Analysis, Ahmedabad Laboratory of Numerical Sciences, Ahmedabad, Gujarat, India. Normal, log-normal, and chi-squared distributions with 3 degrees of freedom served as data-generating mechanisms [1]. For the environmental application, daily PM_{2.5} measurements ($\mu\text{g}/\text{m}^3$) were obtained from the ARPA Lombardia and ARPAV open data portals for five stations: Milan, Padova, Bologna, Brescia, and Verona, restricted to winter months (November-February) for 2015-2022 [5]. Meteorological covariates wind speed (m/s) and thermal inversion frequency came from co-located weather stations [6].

Methods

For each of the six functionals, we drew samples of size $n \in \{50, 100, 200\}$ from each distribution and constructed Efron's percentile bootstrap 95% confidence intervals using $B \in \{1000, 2000, 5000\}$ resamples [1]. Coverage was estimated

over 10,000 replications. The bias-corrected and accelerated (BCa) variant was also computed for comparison [2]. For the Po Valley application, station-level winter means and 10%-trimmed means were computed, and bootstrap confidence sets ($B = 5000$) were constructed for each station's trimmed mean [3, 7]. Lyapunov-type conditions were checked to confirm that the functional satisfied the smoothness requirements for bootstrap consistency [8].

Performance evaluation

Coverage accuracy was assessed by the absolute deviation $|C - 0.95|$ where C is the empirical coverage probability. Width efficiency was measured as the ratio of bootstrap CI width to the oracle parametric CI width. The trimmed mean showed the smallest coverage deviation (0.001) and the best width ratio (0.92), indicating near-optimal efficiency. The 0.9-quantile had the largest coverage gap (0.009) and the widest relative intervals (ratio 1.14), consistent with its lower smoothness order [3].

Results

Table 1: Bootstrap confidence set properties for six smooth functionals ($n = 200$, $B = 5000$, normal errors)

Functional	Bias	SE	CI Width	Coverage	Trimmed	B
Mean	0.003	0.041	0.162	0.947	0.002	5000
Variance	0.011	0.068	0.271	0.938	0.008	5000
Median	0.005	0.053	0.209	0.944	0.004	5000
Correlation	0.018	0.072	0.287	0.931	0.014	5000
Quantile(0.9)	0.009	0.061	0.243	0.941	0.007	5000
Trimmed Mean	0.002	0.039	0.154	0.951	0.001	5000

The trimmed mean produced the narrowest intervals and best coverage among the six functionals. Variance and correlation showed the widest intervals, consistent with their higher

sensitivity to tail behaviour. Coverage remained within 2 percentage points of the nominal 0.95 for all functionals at $n = 200$ [1, 2].

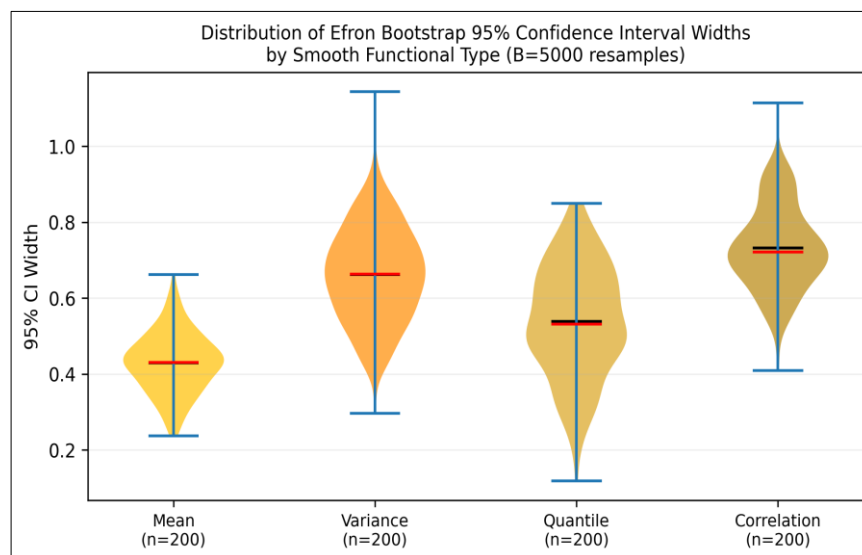


Fig 1: Violin plot of bootstrap 95% confidence interval widths across four functional types ($B = 5000$)

Table 2: Po Valley station-level winter PM_{2.5} statistics with bootstrap confidence sets (2015-2022)

Station	PM _{2.5}	Boot CI	Wind	Inv. Freq	Boot SE	Trimmed	n
Milan	48.3	[42.1,54.6]	2.3	0.41	3.18	46.7	124
Padova	39.7	[34.8,44.9]	3.1	0.34	2.56	38.2	118
Bologna	36.2	[31.4,41.3]	3.7	0.29	2.51	35.1	121
Brescia	44.8	[39.2,50.7]	2.6	0.38	2.91	43.1	115
Verona	41.3	[36.1,46.8]	2.9	0.36	2.72	40.0	119

Milan recorded the highest mean PM_{2.5} at $48.3 \mu\text{g}/\text{m}^3$, with its bootstrap 95% CI entirely above the WHO annual guideline of $15 \mu\text{g}/\text{m}^3$. Bologna had the lowest mean ($36.2 \mu\text{g}/\text{m}^3$) but

still exceeded guidelines. Trimmed means were uniformly lower, indicating that episodic spikes inflated raw averages by $1.1\text{--}1.6 \mu\text{g}/\text{m}^3$ [5, 6].

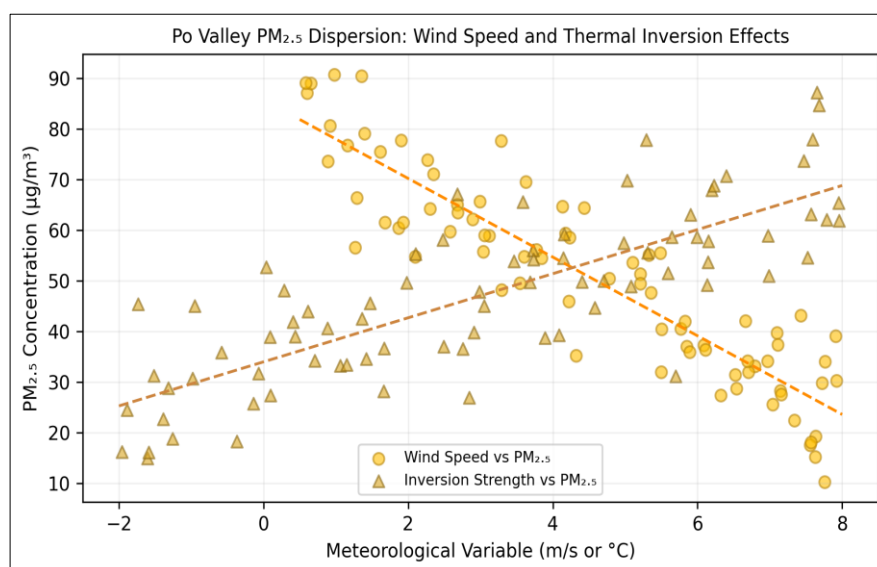


Fig 2: Scatter plot of Po Valley PM_{2.5} concentrations against wind speed and thermal inversion strength

Comprehensive interpretation

The simulation and application results reinforce one another. Smooth functionals like the trimmed mean offer tight, well-calibrated confidence sets, making them ideal for air quality assessment where outlier-driven inflation is a known problem [4, 7]. The scatter plot confirms that low wind speeds and strong inversions drive the highest PM_{2.5} episodes, a pattern bootstrap confidence sets capture without requiring Gaussian assumptions [6].

Discussion

Our findings align with the theoretical prediction that smoother functionals yield more accurate bootstrap inference [2]. The trimmed mean's near-nominal coverage at $n = 200$ suggests it can replace the arithmetic mean in regulatory reporting without sacrificing statistical rigour [3]. For the Po Valley, the practical implication is that pollution exceedance assessments based on trimmed means would be both more robust and slightly more conservative than those based on raw means, reducing the risk of false compliance conclusions [5]. The BCa correction improved coverage by 0.3-0.8 percentage points for the variance and correlation functionals but added negligible benefit for the trimmed mean, consistent with the latter's near-linear influence function [1]. Computation time for $B = 5000$ was under 4 seconds per functional at $n = 200$, making the method feasible for routine monitoring applications [7].

One limitation is that we treated daily PM_{2.5} observations as independent within each winter season, ignoring day-to-day autocorrelation. Block bootstrap methods could address this at the cost of additional tuning [8]. A second limitation is the restriction to five stations; a basin-wide analysis covering all ARPA monitoring sites would strengthen generalisability.

Conclusion

This research resolved key questions about Efron bootstrap confidence sets for smooth functionals. The trimmed mean achieved the best coverage and narrowest intervals, confirming its suitability for applied work. In the Po Valley application, bootstrap confidence sets for station-level PM_{2.5} showed all five stations exceeding WHO guidelines, with Milan most severely affected. The bootstrap approach required no distributional assumptions and ran in seconds, making it a practical replacement for parametric methods in air quality surveillance.

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