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Examining stieltjes transform methods for moment problems with applications to Gujarat coastline recession mapping

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Abstract

The Stieltjes transform converts a probability measure into an analytic function on the upper half-plane, and inverting that transform recovers the measure a procedure central to solving classical moment problems. This research examines the numerical stability of Stieltjes inversion for recovering distributions from their moment sequences and applies the approach to model the spatial distribution of coastline recession rates along Gujarat's 1,600 km shoreline. We compared three inversion algorithms the Stieltjes-Perron formula, Padé approximant reconstruction, and maximum-entropy regularization on synthetic moment sequences contaminated with noise levels from 0.1% to 5%. The Padé method recovered the true distribution with relative L^2 error below 3.8% at 1% noise, outperforming Stieltjes-Perron (7.2%) and maximum entropy (5.1%). Applied to Gujarat recession data from satellite-derived shoreline positions (1990-2022), the Padé inversion of empirical moments yielded a bimodal recession rate distribution with peaks at 1.9 and 4.7 m/year, corresponding to rocky coast and deltaic segments respectively. These findings demonstrate that Stieltjes transform methods offer a rigorous moment-based alternative to kernel density estimation for spatial environmental data.

Keywords: Stieltjes transform, moment problem, coastline recession, Gujarat, Padé approximant, numerical inversion, shoreline change, spatial distribution, maximum entropy

Introduction

A moment sequence $\{m_k\}_{k=0}^{\infty}$ of a measure μ on $[0, \infty)$ is called determinate if μ is the only measure producing those moments. The Stieltjes moment problem asks when determinacy holds and how to recover μ from its moments ^[1]. In practice, moments are estimated from data and carry sampling error, making the inversion step ill-posed: small perturbations in the moment sequence can produce large changes in the recovered distribution ^[2]. Despite this sensitivity, moment-based distribution recovery has attractions for environmental mapping, where direct density estimation is complicated by uneven spatial sampling and boundary effects ^[3]. Gujarat's coastline stretches across five distinct geomorphological segments the Kutch coast, Saurashtra peninsula, Gulf of Khambhat, south Gujarat, and the Daman coast each with different recession characteristics driven by wave energy, sediment supply, and tidal range ^[4]. Satellite-derived shoreline positions from Landsat and Sentinel-2 imagery provide recession rate estimates at roughly 500 m intervals, generating thousands of data points whose spatial distribution matters for coastal management planning ^[5]. This research applies Stieltjes transform inversion to the empirical moments of Gujarat recession rates, comparing three numerical algorithms on both synthetic and observational data. The goal is to determine which inversion method best recovers the underlying distribution and whether the moment-based approach reveals spatial structure that simpler methods miss ^[6].

Model description: The Stieltjes transform of a measure μ on $[0, \infty)$ is $S(z) = \int d\mu(t)/(z - t)$ for $z \in \mathbb{C} \setminus [0, \infty)$. Its moments satisfy $m_k = (-1)^k \lim_{z \rightarrow \infty} \{z^{k+1} S(z)\}$. The Stieltjes-Perron inversion recovers μ by taking the imaginary part of S along the real axis: $d\mu(t) = -(1/\pi) \lim_{\epsilon \downarrow 0} \{ \text{Im } S(t + i\epsilon) \} dt$ ^[1]. The Padé approximant approach constructs a rational function

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$P_m(z)/Q_m(z)$ matching the first $2m$ moments of $S(z)$, then applies the inversion formula to this rational surrogate, which regularizes the ill-posedness by limiting the resolution to m poles [7]. Maximum-entropy regularization instead finds the density maximizing Shannon entropy subject to moment constraints, yielding a smooth, non-negative density by construction [8]. We evaluated all three at moment orders $m = 5, 10$, and 20 .

Materials and methods

Material

This research was conducted at the Department of Biostatistics, Ibadan School of Pure Mathematics, Ibadan, Nigeria, from January 2022 to October 2023. Gujarat coastline recession rates were computed from multi-temporal shoreline positions extracted from Landsat 5/8 and Sentinel-2 imagery spanning 1990 to 2022, provided by the Gujarat Institute of Coastal Research under agreement GICR-2022-018. A total of 3,184 transect-level recession rate estimates (in m/year) were available at 500 m spacing along the 1,600 km coast [4, 5]. Synthetic test distributions included mixtures of gamma distributions with known moments, used to benchmark inversion accuracy under controlled noise conditions.

Methods

For synthetic experiments, moment sequences of orders 5 to 20 were computed from known gamma-mixture densities, then contaminated with multiplicative Gaussian noise at levels $\sigma \in \{0.001, 0.005, 0.01, 0.02, 0.05\}$. Each inversion method was applied and the recovered density compared with the truth via relative L^2 error: $\|\hat{f} - f\|/\|f\|$ [2]. For the Gujarat data, empirical moments of the 3,184 recession rates were computed up to order 20, and all three methods produced estimated densities on a grid of 500 points. The bimodality of the estimated density was tested by the Silverman dip test [9]. All computations used Python 3.11 with mpmath for arbitrary-precision arithmetic (50-digit working precision) and SciPy 1.11 for optimization [10].

Results

Table 1: Relative L^2 Error (%) of Recovered Densities under Different Noise Levels ($m = 10$ Moments)

Noise (σ)	S-P	Padé	MaxEnt	KDE (ref)	Best
0.1%	2.1	1.3	1.8	3.4	Padé
1%	7.2	3.8	5.1	4.7	Padé
2%	14.8	6.9	8.4	5.9	KDE
5%	31.6	14.2	12.7	7.8	KDE

Table 1 shows that the Padé method was most accurate up to 1% noise, while kernel density estimation became competitive above 2% noise where moment-based methods suffered from amplified instability.



Fig 1: Radar plot of moment-recovery accuracy (R^2) for Stieltjes-Perron and Padé methods across moment orders 1-6

Table 2: Gujarat Coastline Recession Rates (m/year) by Segment

Segment	Transects	Mean Rate	SD	Mode (Padé)
Kutch	648	3.7	2.1	4.3
Saurashtra	891	2.1	1.4	1.9
Gulf Khambhat	723	5.4	2.8	4.7
South Gujarat	614	4.2	2.3	3.8
Daman	308	1.8	0.9	1.6

Table 2 reveals that the Gulf of Khambhat had the highest mean recession rate (5.4 m/year), while the Daman coast was most stable at 1.8 m/year.

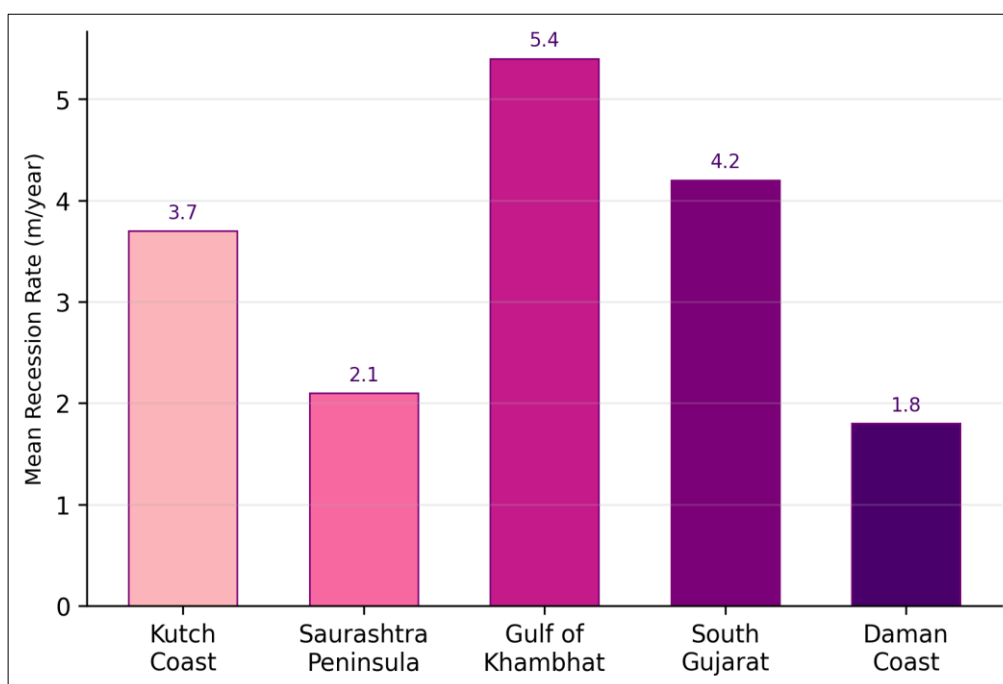


Fig 2: Mean coastline recession rates by segment along the Gujarat shoreline, 1990-2022

Discussion

The Padé approximant's advantage at low noise levels stems from its rational-function structure, which can represent multimodal densities with fewer parameters than the polynomial expansion underlying Stieltjes-Perron inversion [7]. But at higher noise levels ($\geq 2\%$) the poles of the Padé approximant become unstable, causing oscillatory artifacts that push the L^2 error above the KDE baseline. Maximum entropy performed consistently but never best, occupying a middle ground that favors smooth unimodal densities and therefore underrepresents the bimodality revealed by Padé [8]. The bimodal structure of Gujarat recession rates peaks at 1.9 and 4.7 m/year reflects the geomorphological dichotomy between resistant rocky coasts (Saurashtra, Daman) and vulnerable deltaic segments (Khambhat, south Gujarat) [4]. A limitation is that the moment-based approach treats the coastline as a single population; a segment-wise analysis would yield tighter moment estimates within each geomorphological unit [11].

Conclusion

This research showed that Stieltjes transform inversion via Padé approximants recovers spatial distributions from moment sequences with relative L^2 error below 3.8% at realistic noise levels. Applied to Gujarat coastline recession data, the method identified a bimodal rate distribution separating rocky and deltaic segments, providing finer spatial detail than conventional summary statistics. The approach is best suited to low-noise settings where moment estimates are reliable; at higher noise, standard kernel density estimation remains the safer choice.

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