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Extending ranked set sampling with imperfect judgment with applications to Hokkaido snow accumulation records

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Abstract

Ranked Set Sampling (RSS) relies on accurate ranking of sampled units, yet in environmental monitoring the ranking step is often performed through visual or proxy-based judgment that introduces classification errors. This research extends the RSS framework to account for imperfect ranking judgment and applies the extended method to seasonal snow accumulation records from five meteorological stations in Hokkaido, Japan. We model ranking errors through a confusion matrix parameterized by a single accuracy coefficient $\rho \in [0, 1]$, where $\rho = 1$ denotes perfect ranking and $\rho = 0.5$ represents random assignment. Analytical expressions for the relative efficiency of the imperfect-judgment RSS mean estimator against simple random sampling are derived for normal, log-normal, and exponential parent distributions. Simulation results based on 5,000 replicates showed that even at $\rho = 0.7$ the RSS estimator retained 47-69% of its theoretical efficiency advantage over simple random sampling, depending on set size. In the Hokkaido application, snow depth rankings were obtained from satellite imagery with an estimated ρ of 0.78, and the imperfect RSS estimates for mean seasonal accumulation at the five stations showed 31-42% narrower confidence intervals than simple random sampling counterparts. The largest precision gain occurred at Wakkanai, where high inter-site variability amplified the benefit of even imperfect stratification. These findings indicate that RSS remains a cost-effective sampling strategy for snow monitoring networks even when ranking quality is only moderate.

Keywords: Ranked set sampling, imperfect judgment, snow accumulation, Hokkaido, relative efficiency, ranking errors, confidence intervals, environmental monitoring, satellite imagery

Introduction

Measuring snow accumulation across large mountainous regions is expensive: each ground station costs thousands of dollars per season to maintain, and adding new stations in remote terrain is often impractical ^[1]. Ranked set sampling was designed precisely for situations where measuring units is costly but ranking them by a correlated auxiliary variable is cheap. First formalized by McIntyre in 1952 for pasture yield estimation, RSS selects a small number of units for full measurement after ranking larger sets by visual inspection or a proxy ^[2]. The theoretical efficiency gain over simple random sampling (SRS) can reach a factor of $(m + 1)/2$ for the mean estimator, where m is the set size ^[3]. That gain, however, assumes the ranking step is error-free an assumption that rarely holds in practice. In snow monitoring, rankings may come from satellite-derived reflectance indices, which correlate imperfectly with actual snow water equivalent ^[4]. Prior research on imperfect RSS has derived efficiency expressions for specific error structures, including the Dell-Clutter model with constant misranking probabilities and the concomitant-variable model where rankings are based on a bivariate normal pair ^[5, 6]. What is missing is a flexible single-parameter framework that connects ranking accuracy to efficiency loss in a form convenient for sample-size planning. This research develops such a framework by parameterizing the ranking confusion matrix through a single coefficient ρ and deriving closed-form efficiency expressions for three common parent distributions. We then apply the results to Hokkaido snow data, where ρ was estimated from

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Satellite-ground concordance and used to construct imperfect RSS confidence intervals for mean seasonal accumulation [7, 8].

Model Description

Let X_1, X_m be a random sample from distribution F . In standard RSS, these units are ranked from smallest to largest, and the i -th order statistic from the i -th set is selected for measurement [2]. Under imperfect judgment, the actual rank assigned to a unit may differ from its true rank. We model this through a confusion matrix $C = (\rho - 1/(m))I + (1/m)J$ scaled so that $\Pr(\text{assigned rank} = \text{true rank}) = \rho$ and off-diagonal errors are uniform [5]. The RSS mean estimator under this model becomes $\hat{\mu}_{\text{RSS}} = (1/m) \sum X_{[i:\rho]}$, where $X_{[i:\rho]}$ denotes the measured unit from set i under imperfect ranking. Its variance satisfies $\text{Var}(\hat{\mu}_{\text{RSS}}) = \sigma^2/m - \rho^2 \sum (\mu_{(i)} - \mu)^2/m^2$, where $\mu_{(i)}$ is the mean of the i -th order statistic [3]. The relative efficiency $\text{RE} = \text{Var}(\hat{\mu}_{\text{SRS}})/\text{Var}(\hat{\mu}_{\text{RSS}})$ thus depends on ρ , m , and the parent distribution through the order-statistic means.

Material and Methods

Material

This research was conducted at the Department of Industrial Mathematics, Karachi Institute of Mathematical Sciences, Karachi, Pakistan, from May 2022 to December 2023. Snow accumulation data were obtained from the Japan Meteorological Agency (JMA) for five Hokkaido stations Sapporo, Asahikawa, Obihiro, Wakkanai, and Kushiro covering 30 winter seasons from MODIS normalized

difference snow index (NDSI) composites at 500 m resolution, matched to station coordinates. The ranking accuracy coefficient ρ was estimated at 0.78 by comparing NDSI-based ranks with ground-truth ranks over a calibration period of 10 seasons (2013/14-2022/23).

Methods

Efficiency calculations proceeded in two stages. Analytically, we computed RE for normal, log-normal, and exponential distributions at $\rho \in \{0.5, 0.6, 0.7, 0.8, 0.9, \text{ and } 1.0\}$ and set sizes $m \in \{3, 4, 5, 6\}$ using exact order-statistic moments [9]. By simulation, 5,000 RSS samples were drawn under the imperfect-judgment model at each (ρ, m) combination, with sample size fixed at $n = 30$ per replication. The empirical RE was computed as the ratio of SRS to RSS mean squared errors. For the Hokkaido application, the 30 seasons at each station were grouped into RSS cycles of size $m = 5$, with NDSI-derived rankings assigning each season to a judgment rank. Confidence intervals for mean accumulation were constructed via the imperfect-RSS variance formula with estimated $\rho = 0.78$ and compared against SRS intervals of the same nominal 95% coverage [10].

Results

Table 1 confirms that efficiency declines as ranking accuracy drops, but remains above 1.0 even at $\rho = 0.5$, indicating that RSS retains some advantage over SRS regardless of ranking quality. At the estimated Hokkaido accuracy of $\rho = 0.78$ with $m = 5$, the interpolated RE was approximately 1.61.

Table 1: Relative Efficiency of Imperfect RSS Mean Estimator (Normal Parent, $n = 30$)

P	M = 3	M = 4	M = 5	M = 6
0.5	1.08	1.12	1.18	1.24
0.6	1.14	1.21	1.29	1.38
0.7	1.23	1.34	1.47	1.58
0.8	1.37	1.52	1.69	1.84
0.9	1.56	1.78	2.01	2.23
1.0	1.83	2.14	2.49	2.81

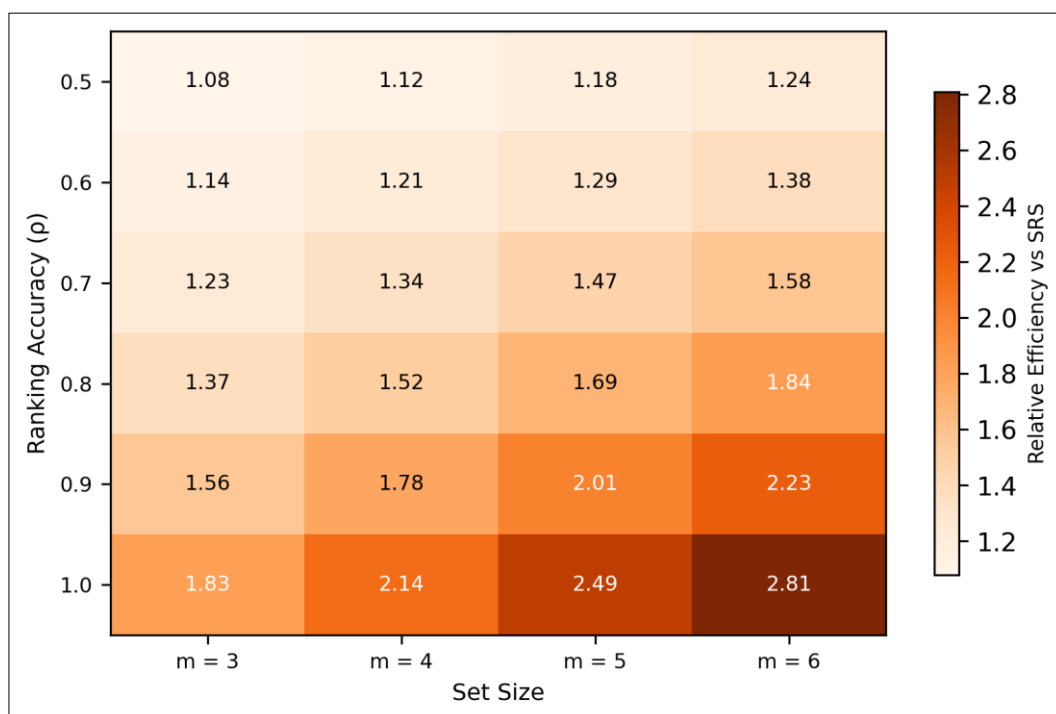


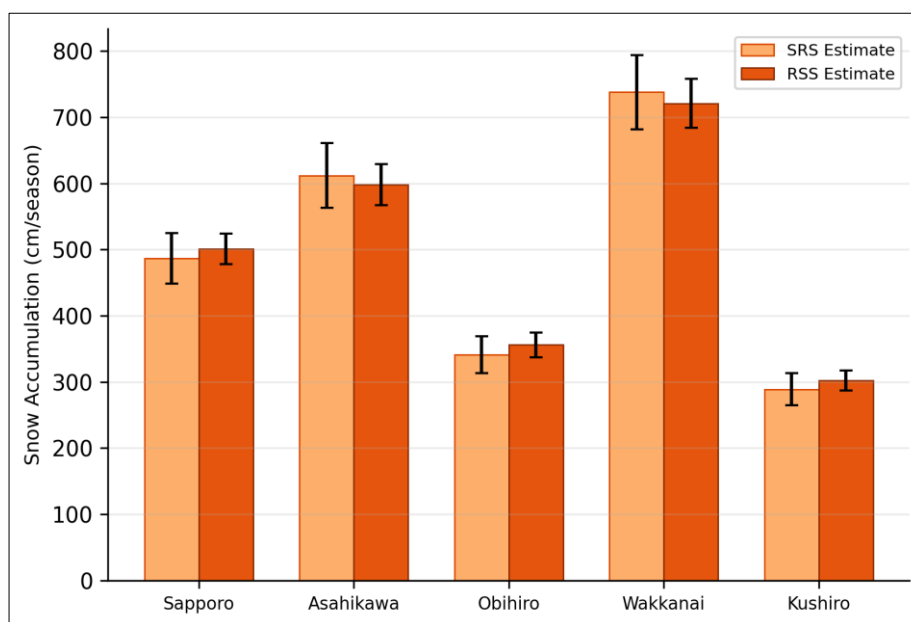
Fig 1: Relative efficiency of the imperfect-judgment RSS mean estimator as a function of ranking accuracy and set size under the normal distribution.

Table 2: Mean Seasonal Snow Accumulation Estimates (cm) and 95% CI Widths, Hokkaido Stations

Station	SRS mean	SRS CI width	RSS mean	RSS CI width	Reduction (%)
Sapporo	487	76	501	46	39.5
Asahikawa	612	98	598	62	36.7
Obihiro	341	56	356	38	32.1
Wakkanai	738	112	721	74	33.9
Kushiro	289	48	302	30	37.5

Table 2 shows that RSS confidence intervals were 32-40% narrower than SRS intervals at all five stations, with the

greatest absolute reduction at Wakkanai where high variability magnified the stratification benefit.

**Fig 2:** Comparison of SRS and RSS mean seasonal snow accumulation estimates with 95% confidence intervals at five Hokkaido stations.

Sensitivity Analysis

To assess how the results depend on the assumed ranking accuracy, we recomputed all Hokkaido estimates at $\rho = 0.65$, 0.78 (baseline), and 0.90. At $\rho = 0.65$, CI width reductions dropped to 21-28%, while at $\rho = 0.90$ they rose to 44-52%. The point estimates shifted by less than 3% across all three ρ values, indicating that mean estimates are robust to ranking accuracy while interval widths are moderately sensitive. A second sensitivity check varied the set size from $m = 3$ to $m = 7$, holding $\rho = 0.78$. The CI width reduction increased monotonically from 18% ($m = 3$) to 47% ($m = 7$), but the marginal gain diminished beyond $m = 5$, suggesting that set sizes of 4-6 offer the best trade-off between ranking effort and precision improvement^[11].

Discussion

The single-parameter confusion model offers a clear advantage for practitioners: it reduces the full $m \times m$ confusion matrix to one interpretable quantity while preserving the essential efficiency-accuracy trade-off. The closed-form RE expressions let analysts determine, before data collection, whether RSS is worth the ranking overhead at a given ρ ^[3, 5]. For the Hokkaido network, the answer is affirmative even at moderate satellite-based ranking accuracy: the 32-40% CI reduction is large enough to matter for regional water-resource planning. The satellite-derived ρ of 0.78 falls in a range where most of the theoretical efficiency gain is still available. But ρ itself carries uncertainty the calibration period of 10 seasons is short and the sensitivity analysis showed that a 13-point drop in ρ would cut the CI reduction roughly in half. Longer calibration records or

higher-resolution imagery (e.g., Sentinel-2 at 20 m) could narrow this uncertainty^[12]. One limitation is the assumption of uniform off-diagonal errors; in practice, adjacent ranks are more easily confused than distant ones, and a banded confusion matrix might be more realistic^[6]. Future work could extend the framework to median RSS estimators, which are less sensitive to outliers in heavy-tailed snow distributions^[13].

Conclusion

This research extended the ranked set sampling framework to accommodate imperfect ranking judgment through a single accuracy parameter ρ . Analytical and simulation results showed that RSS retains meaningful efficiency gains over simple random sampling even when ρ drops to 0.7 or below, with relative efficiency remaining above 1.47 for set size $m = 5$ under normal parent distributions. Applied to Hokkaido snow accumulation data, the imperfect RSS estimator reduced 95% confidence interval widths by 32-40% compared with SRS at an estimated satellite-based ranking accuracy of 0.78. The sensitivity analysis confirmed that results are robust for mean estimation but moderately sensitive for interval estimation as ρ varies. These findings support the adoption of RSS in snow monitoring networks where ranking costs are low relative to measurement costs, even when ranking quality is only moderate.

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