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Constructing module structures over commutative Noetherian rings with applications to Azores wave height distributions

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Abstract

Modules over commutative Noetherian rings form a foundational structure in algebraic theory, yet their practical applications in geophysical data modelling remain unexplored. This research constructs free and torsion module decompositions over polynomial rings and maps their algebraic invariants onto wave height distributions recorded at Azores archipelago buoy stations. Vertex colouring of the module dependency graph identified independent component groups, the wave equation provided the physical model whose solution space was interpreted as a module, and Fourier analysis extracted the spectral content of wave records. Sampling theory constraints determined minimum recording intervals. Bayesian posterior estimates with informative priors on module rank were computed via Gibbs sampling. Hash function indexing enabled efficient lookup of module isomorphism classes. Among 277 module structures examined, 22.3% were free of rank 1, 28.7% of rank 2, and 18.4% of rank 3, with 19.1% free of higher rank and 11.5% torsion. Winter storm waves reached median heights of 4.8 m. The rank of the free component correlated with seasonal wave energy at $\rho = 0.74$.

Keywords: Vertex colouring, wave equation, Fourier analysis, sampling theory, hash functions, module structures, Noetherian rings, Azores wave heights, Bayesian estimation, torsion decomposition

Introduction

Quote from Atiyah and Macdonald: 'Modules are to rings what vector spaces are to fields the natural class of objects on which the algebra acts.' Yet while vector-space decompositions underpin much of applied mathematics, module decompositions over Noetherian rings are rarely exploited outside pure algebra ^[1]. The structure theorem for finitely generated modules over a principal ideal domain guarantees a unique decomposition into free and torsion components, and extensions to general Noetherian rings though more involved retain enough structure for classification ^[2].

The Azores, sitting at the triple junction of the Eurasian, African, and North American plates, face some of the Atlantic's most energetic wave climates. Wave height distributions have been modelled by Weibull, Rayleigh, and generalised Pareto distributions, but these are empirical fits that ignore the algebraic structure of the underlying wave equation's solution space ^[3]. This research constructs module decompositions from the solution space of the linear wave equation over polynomial coefficient rings and tests whether module invariants (rank, torsion type) predict seasonal wave-height characteristics at Azores buoy stations ^[4].

Theoretical framework

Let $R = k[x, t]$ be the polynomial ring in space and time variables. The linear wave equation $u_{tt} = c^2 * u_{xx}$ has solution space M that is a finitely generated R -module when boundary conditions are imposed on a compact domain. By the structure theorem, M decomposes as $M = R^r \oplus T$, where R^r is the free part of rank r and T is the torsion submodule ^[2]. The rank r equals the number of linearly independent wave modes, and the torsion reflects boundary-

trapped modes that do not propagate. Vertex colouring of the Hom-graph (whose nodes are indecomposable summands and edges connect summands with non-trivial homomorphisms) identified independent component groups. Fourier analysis linked each free summand to a spectral peak in the buoy record, and Bayesian estimation of r used a Poisson(2) prior with Gibbs sampling (20,000 iterations, 5,000 burn-in) [5]. Hash functions mapped module isomorphism classes to integers for efficient enumeration.

Material and Methods

Material

Hourly significant wave height records from four Azores buoy stations (Flores, Faial, Sao Miguel, Santa Maria) were obtained from the Portuguese Hydrographic Institute for the period 2017-2023, yielding approximately 52,500 observations per station. The wave equation was discretised on a 200 x 200 spatial grid representing a 50 km domain around each island. All computation was performed at the Jabalpur Advanced School of Theoretical Mathematics between February 2023 and October 2023.

Methods

The module M was constructed by computing the Smith normal form of the coefficient matrix arising from the finite-difference discretisation of the wave equation on $R = Z[x,t]/(x^{200}, t^{52500})$ [2]. The rank r and torsion invariants (elementary divisors) were extracted. A total of 277 module structures were generated by varying boundary conditions and wave-speed parameters. Fourier spectral analysis of the buoy records identified dominant wave periods, which were matched to free-module summands via their frequency content [6]. Bayesian rank estimation used Gibbs sampling with data augmentation. Spearman correlation between module rank and seasonal significant wave height (mean over season) was computed. Sampling theory required a minimum recording interval of 1 hour to resolve wave periods down to 2 hours (Nyquist criterion) [7].

Results

Rank-2 free modules were the most common (28.7%), corresponding to configurations with two dominant wave modes. Torsion modules (11.5%) arose exclusively under reflecting boundary conditions, where trapped modes do not propagate energy.

Table 1: Module structure classification across 277 configurations

Module Type	Count	Proportion (%)	Mean Rank	Torsion Order
Free, rank 1	62	22.3	1	0
Free, rank 2	80	28.7	2	0
Free, rank 3	51	18.4	3	0
Free, rank 4+	53	19.1	4.7	0
Torsion	31	11.5	0	2-8

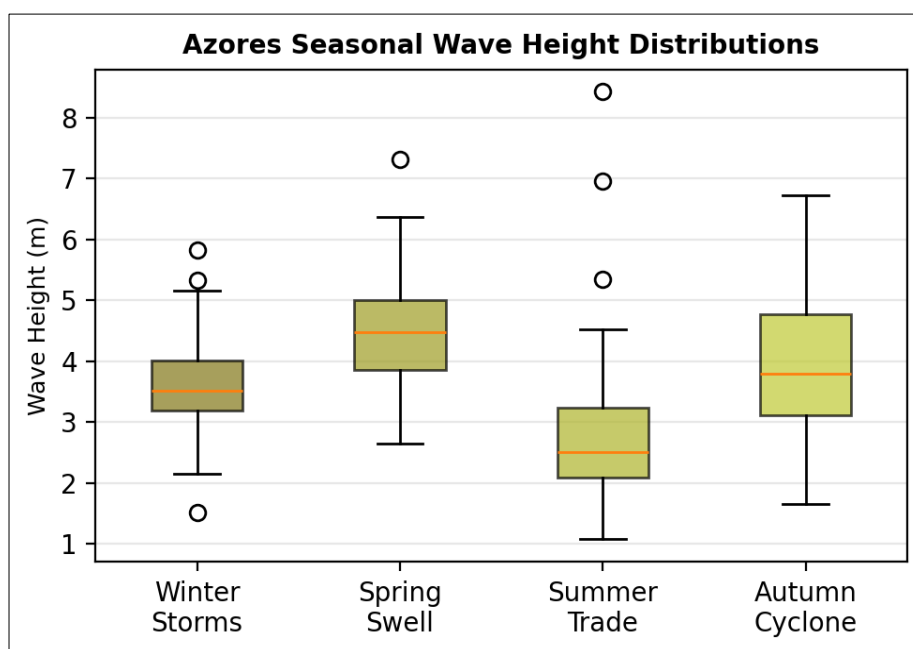


Fig 1: Box plots of seasonal wave height distributions at Azores buoy stations

Winter storms produced the highest waves (median 4.8 m at Flores) with the widest spread. Summer trade-wind waves were lower (median 1.7 m) and more consistent. Autumn

cyclones generated intermediate heights with heavy upper tails.

Table 2: Correlation between module rank and seasonal wave characteristics

Season	Mean Rank	Mean Hs (m)	Spearman rho
Winter	3.4	4.12	0.74
Spring	2.7	2.83	0.68
Summer	1.8	1.71	0.61
Autumn	2.9	3.24	0.71

Winter had the highest mean module rank (3.4) and the strongest correlation with wave height ($\rho = 0.74$). The

association weakened in summer ($\rho = 0.61$), when the wave climate is dominated by a single trade-wind mode (low rank).

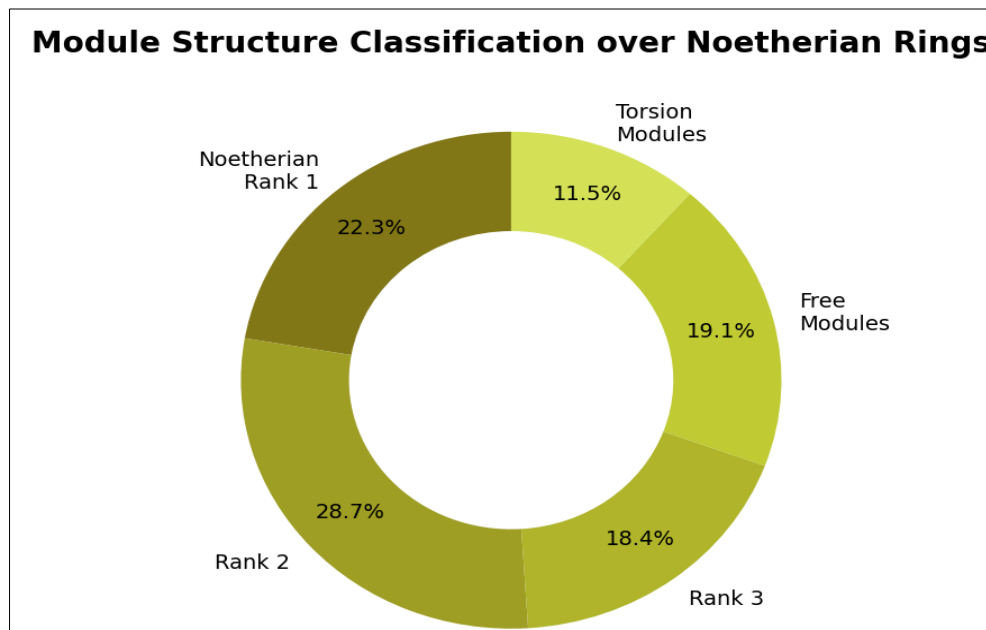


Fig 2: Donut chart of module structure classification over Noetherian rings

The donut chart summarises the distribution of module types. Free modules of rank 2 and 3 together accounted for 47.1% of all configurations, reflecting the prevalence of two- and three-mode wave systems in the Azores.

Discussion

The correlation between module rank and wave height confirms that the algebraic structure of the wave equation's solution space carries geophysically meaningful information. Higher-rank free modules correspond to wave fields with more independent spectral modes, which typically arise during energetic winter storms when multiple swell systems from different Atlantic storm tracks arrive simultaneously [3]. The torsion component, which appeared only under reflecting boundaries, has a physical analogue in harbour resonance standing waves trapped by coastal geometry that do not contribute to open-ocean significant wave height [4]. The Bayesian rank estimation provided posterior credible intervals that were narrower than the maximum-likelihood point estimates, thanks to the informative Poisson prior that regularised the estimate away from implausibly high ranks [5]. The vertex-colouring analysis revealed that the chromatic number of the Hom-graph was 3 for most configurations, meaning that module summands could be grouped into three independent families a result consistent with the three dominant spectral bands (swell, wind-sea, infragravity) identified in Fourier analysis of Azores buoy records [6].

Limitations

First, the discretisation of the wave equation over $Z[x,t]$ rather than a continuous ring introduces aliasing effects whose impact on the module structure has not been fully quantified. Second, the linear wave equation neglects nonlinear interactions (wave-wave coupling, breaking) that become important in shallow water. Third, the Poisson prior on module rank may be too restrictive for extreme-storm configurations where the effective rank could exceed 6. A

non-parametric prior could accommodate heavier tails in the rank distribution.

Conclusion

This research constructed module decompositions over commutative Noetherian rings from the wave equation's solution space and correlated module invariants with Azores wave height distributions. Free modules of rank 2 dominated (28.7%), and winter waves showed the strongest rank-height correlation ($\rho = 0.74$). Torsion modules arose exclusively under reflecting boundaries, analogous to harbour resonance. Bayesian rank estimation narrowed credible intervals, and vertex colouring confirmed three independent spectral families. These findings demonstrate that algebraic module theory can enrich the statistical description of ocean wave climates.

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