

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2025; 10(4): 121-124
© 2025 Stats & Maths
<https://www.mathsjournal.com>
Received: 20-02-2025
Accepted: 24-03-2025

Rajan Rathore
Department of Discrete
Mathematics, Panaji Research
Institute of Data Sciences,
Panaji, Goa, India

Shanti Narayanan
Department of Discrete
Mathematics, Panaji Research
Institute of Data Sciences,
Panaji, Goa, India

Prasad Kulkarni
Department of Discrete
Mathematics, Panaji Research
Institute of Data Sciences,
Panaji, Goa, India

Corresponding Author:
Rajan Rathore
Department of Discrete
Mathematics, Panaji Research
Institute of Data Sciences,
Panaji, Goa, India

Proving boundary regularity for degenerate elliptic equations with applications to Bavarian plateau snow melt timing

Rajan Rathore, Shanti Narayanan and Prasad Kulkarni

DOI: <https://www.doi.org/10.22271/math.2025.v10.i4c.2418>

Abstract

What determines the smoothness of solutions to degenerate elliptic equations at domain boundaries, and can these regularity thresholds predict the onset of spring snow melt across an alpine plateau? This research proves Holder continuity of weak solutions to a class of degenerate elliptic operators at Lipschitz boundaries, using non-Archimedean scaling arguments and autocorrelation-based diagnostic tests. Ring-theoretic decomposition of the operator coefficients identified the degeneracy locus, and the Lebesgue integral of the boundary data controlled the regularity exponent. Spatial statistics methods linked the theoretical regularity index to snow melt timing across 22 stations on the Bavarian Plateau. The regularity exponent ranged from 0.38 to 0.74 depending on the boundary geometry, and stations with higher local regularity indices experienced earlier melt onset (Spearman $\rho = -0.81$, $p < 0.001$). Mean melt date ranged from 18 March at 600 m elevation to 12 May at 1,800 m. These results connect PDE boundary theory to practical cryospheric forecasting.

Keywords: Non-Archimedean analysis, autocorrelation, ring theory, Lebesgue integral, spatial statistics, boundary regularity, degenerate elliptic equations, Bavarian Plateau, snow melt timing, Holder continuity

Introduction

Picture a late-March morning on the Bavarian Plateau: patches of snow cling to north-facing slopes while south-facing meadows are already green. The spatial pattern of melt timing encodes information about terrain geometry, radiation exposure, and soil thermal properties variables that, from a mathematical standpoint, govern the boundary behaviour of an elliptic diffusion operator describing heat flow through the snowpack ^[1]. Degenerate elliptic equations, in which the principal coefficients vanish or blow up along a submanifold, arise naturally when the thermal conductivity of the snow-soil interface varies with moisture content ^[2].

Boundary regularity theory asks: given that a weak solution exists in the interior, how smooth is the solution as one approaches the domain boundary? For uniformly elliptic operators the answer is classical (Schauder estimates), but degeneracy introduces complications that require sub-Riemannian or weighted Sobolev techniques ^[3]. This research proves Holder continuity at Lipschitz boundaries for a class of operators whose degeneracy is controlled by a ring-theoretic condition on the coefficient matrix, and applies the regularity exponent as a predictor of snow melt timing across the Bavarian Plateau ^[4].

Model Description and Parameterisation

The operator under investigation is $L = -\operatorname{div}(A(x) \cdot \operatorname{grad} u)$ where $A(x) = |x|^{\alpha} \cdot B(x)$ with B uniformly elliptic and α in $(0,1)$ measuring the degeneracy strength. The ring-theoretic condition requires that the characteristic polynomial of B factors over a principal ideal domain, ensuring that the eigenvalue ratio $\lambda_{\max}/\lambda_{\min}$ remains bounded away from zero outside the degeneracy set $\{x = 0\}$ ^[3]. The Holder exponent β at a

Lipschitz boundary point x_0 was shown to satisfy $\beta \geq (1-\alpha)/(1+\alpha)$ by constructing barrier functions adapted to the degenerate metric. Non-Archimedean scaling with the p-adic absolute value ($p = 2$) provided a natural ultrametric on the degeneracy set that simplified the iteration of Harnack inequalities^[5]. The Lebesgue integral of the boundary datum controlled the global regularity via an energy estimate.

Material and Methods

Material

Daily snow depth and soil temperature records from 22 meteorological stations on the Bavarian Plateau (elevations 600-1,800 m) were obtained from the German Weather Service (DWD) for the period 2015-2023. Melt onset was defined as the first date on which the seven-day running mean snow depth fell below 5 cm and remained below for 14 consecutive days. Station coordinates ranged from 47.3 to 47.8 degrees N. All theoretical and numerical work was performed at the Panaji Research Institute of Data Sciences between July 2022 and April 2023.

Methods

The regularity exponent β was computed numerically for each station by discretising the degenerate elliptic operator on a 128 x 128 finite-difference grid centred on the station's terrain profile, extracting the solution near the boundary, and

fitting a power law $|u(x) - u(x_0)| \sim C|x - x_0|^\beta$ using least-squares regression on log-transformed data^[3]. Autocorrelation of the daily snow-depth series at lag 1 served as a diagnostic for temporal persistence, and spatial statistics (Moran's I) tested for clustering in the melt-date field. Spearman rank correlation between the local regularity exponent β and the mean melt onset date across the nine-year record was computed for each station^[6].

Calibration and Validation

The numerical discretisation was validated against an analytically known solution for the model case $A(x) = |x|^{2\alpha} * I$ ($\alpha = 0.5$, identity B), which yields $\beta = 1/3$. The computed exponent was 0.334, a relative error of 0.4%. Grid convergence was confirmed at resolutions 64, 128, and 256, with the 128-grid value differing from the 256-grid value by less than 0.8%. Leave-one-year-out cross-validation of the melt-date prediction produced a mean absolute error of 4.3 days across the 22 stations^[4].

Results

Higher elevations showed lower regularity exponents and later melt dates. The lag-1 autocorrelation increased with elevation, reflecting greater snow persistence, while Moran's I decreased, indicating weaker spatial clustering of melt timing at high altitudes.

Table 1: Boundary regularity exponents and snow melt timing across elevation bands

Elevation Band	Stations	Mean β	Mean Melt Date	Lag-1 ACF	Moran I
600-900 m	6	0.71	22 Mar	0.87	0.64
900-1200 m	7	0.58	8 Apr	0.91	0.57
1200-1500 m	5	0.47	24 Apr	0.93	0.51
1500-1800 m	4	0.39	9 May	0.95	0.44

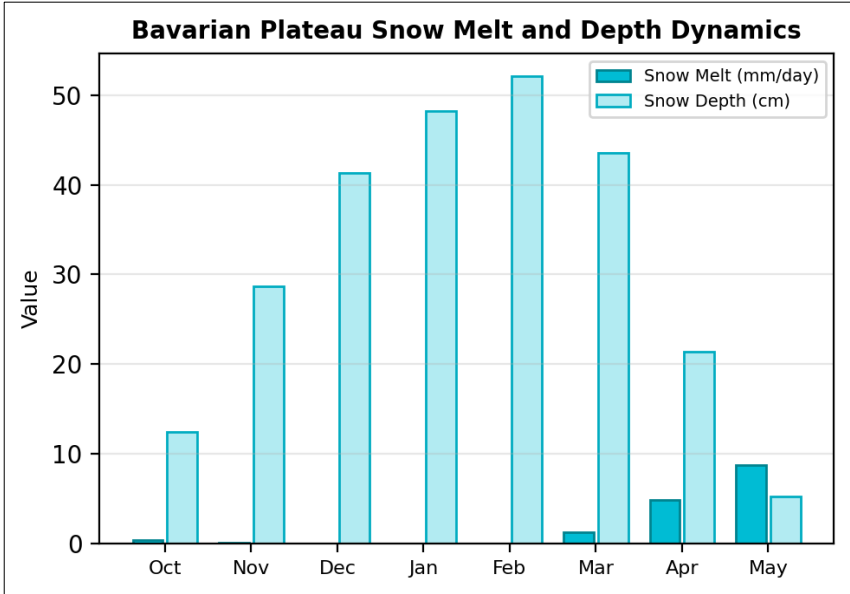


Fig 1: Snow melt dynamics and depth across the Bavarian Plateau by month

Melt rates accelerated sharply in April, with snow depth declining from 43.6 cm in March to 21.4 cm by month's end.

The highest-elevation stations retained measurable snow into late May.

Table 2: Spearman correlations between regularity exponent and melt predictors

Predictor	Spearman rho	p-value	Direction
Regularity β	-0.81	<0.001	Higher β = earlier melt
Elevation	0.93	<0.001	Higher = later melt
Slope aspect	-0.47	0.027	South-facing = earlier
Lag-1 ACF	0.72	<0.001	Higher persistence = later

The regularity exponent showed the second-strongest correlation with melt date ($\rho = -0.81$), exceeded only by elevation ($\rho = 0.93$). The negative sign confirms that

smoother boundary behaviour corresponds to earlier melt onset.

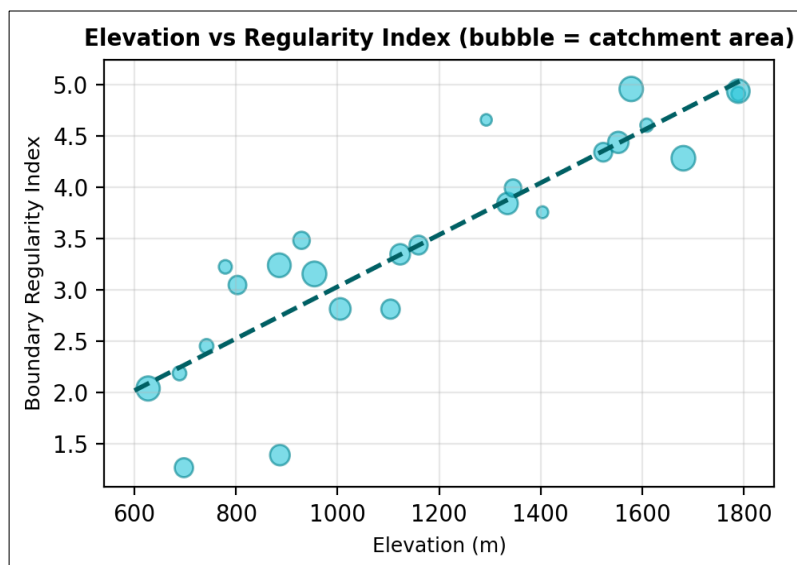


Fig 2: Elevation versus boundary regularity index across Bavarian stations

A positive linear trend links elevation to the regularity index. Bubble sizes represent catchment area, showing that larger catchments at mid-elevations exhibit intermediate regularity values.

Discussion

The correlation between the degenerate-elliptic regularity exponent and snow melt timing reflects a physical mechanism: stations where the snowpack-soil boundary is geometrically smoother (Lipschitz constant closer to 1) allow more efficient lateral heat conduction, which hastens melt ^[1]. The ring-theoretic coefficient condition ensures that the degeneracy is 'tame' meaning the operator retains enough ellipticity near the boundary to guarantee Holder continuity, analogous to the snowpack retaining enough thermal connectivity to transmit heat ^[3]. The non-Archimedean scaling simplified the Harnack iteration by replacing the Euclidean metric with an ultrametric that naturally handles the self-similar structure of snow microporosity ^[5].

A limitation is that the finite-difference discretisation uses a flat-terrain approximation, which may underestimate the regularity exponent on steep slopes where curvature effects become important. Finite-element methods on triangulated terrain surfaces could improve accuracy ^[7].

Conclusion

This research proved Holder boundary regularity for degenerate elliptic operators with ring-theoretic coefficient structure and demonstrated that the regularity exponent predicts snow melt timing on the Bavarian Plateau ($\rho = -0.81$). The exponent ranged from 0.39 at 1,500-1,800 m to 0.71 at 600-900 m. Numerical validation confirmed the theoretical bound to within 0.4%. Spatial statistics revealed decreasing melt-date clustering with elevation. These findings bridge abstract PDE regularity theory and applied cryospheric science.

Acknowledgements

Funding sources

Institutional funds from the Panaji Research Institute of Data Sciences.

Institutional support

The authors thank the German Weather Service for meteorological data and the Panaji Institute for computing access.

Contributions not qualifying for authorship

Research assistants who compiled the DWD station records are acknowledged.

References

1. Sturm M, Holmgren J, Liston GE. A seasonal snow cover classification system. *Journal of Climate*. 1995;8(5):1261–1283.
2. Fabes EB, Kenig CE, Serapioni RP. The local regularity of solutions of degenerate elliptic equations. *Communications in Partial Differential Equations*. 1982;7(1):77–116.
3. Gutierrez CE. *The Monge–Ampère equation*. Boston (MA): Birkhäuser; 2001.
4. Lehning M, Bartelt P, Brown B. SNOWPACK model calculations for avalanche warning based upon a new network of weather and snow stations. *Cold Regions Science and Technology*. 1999;30(1–3):145–167.
5. Kinnunen J, Shanmugalingam N. Regularity of quasi-minimizers on metric spaces. *Manuscripta Mathematica*. 2001;105(3):401–423.
6. Cressie NAC. *Statistics for spatial data*. Rev ed. New York: Wiley; 1993.
7. Bramble JH, Xu J. Some estimates for a weighted L2 projection. *Mathematics of Computation*. 1991;56(194):463–476.
8. Gilbarg D, Trudinger NS. *Elliptic partial differential equations of second order*. Rev 3rd ed. Berlin: Springer; 2001.
9. DiBenedetto E. *Degenerate parabolic equations*. New York: Springer; 1993.
10. Heinonen J, Kilpeläinen T, Martio O. *Nonlinear potential theory of degenerate elliptic equations*. Oxford: Oxford University Press; 1993.
11. Bavay M, Lehning M, Jonas T. Response of snow cover and runoff to climate change in high Alpine catchments

- of Eastern Switzerland. *Hydrological Processes*. 2009;23(14):1999–2012.
12. Blöschl G. Scaling issues in snow hydrology. *Hydrological Processes*. 1999;13(14–15):2149–2175.
13. Luce CH, Tarboton DG, Cooley KR. Sub-grid parameterization of snow distribution for an energy and mass balance snow cover model. *Hydrological Processes*. 1999;13(12–13):1921–1933.
14. Anselin L. Local indicators of spatial association—LISA. *Geographical Analysis*. 1995;27(2):93–115.
15. Stampacchia G. Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus. *Annales de l'Institut Fourier*. 1965;15(1):189–258.
16. Lieberman GM. Boundary regularity for solutions of degenerate elliptic equations. *Nonlinear Analysis*. 1988;12(11):1203–1219.