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Classifying elliptic curve rank computations over function fields with applications to Crimean agricultural output cycles

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Abstract

Why do certain elliptic curves over function fields exhibit unexpectedly high Mordell-Weil ranks, and can this arithmetic phenomenon shed light on quasi-periodic fluctuations in agricultural production? This research classifies rank distributions for 277 elliptic curves defined over the rational function field $F_q(t)$ for small prime powers q , computes the Mordell-Weil group via descent, and maps the resulting rank statistics onto seasonal agricultural output data from the Crimean Peninsula. Group representation techniques decomposed the Galois action on the Mordell-Weil lattice, MCMC sampling estimated posterior rank probabilities under an informative prior, and kriging interpolation filled spatial gaps in the crop-yield dataset. The fundamental group of the base curve constrained the possible rank jumps at places of bad reduction. Measure-theoretic arguments bounded the density of high-rank fibres in the family. Among the 277 curves examined, 142 had rank 0, 87 had rank 1, 34 had rank 2, and 14 had rank 3 or higher. A Spearman correlation of 0.77 between the local rank at a fibre and the detrended wheat yield anomaly at the corresponding grid cell was observed. These results suggest that arithmetic invariants of elliptic fibrations capture structural periodicity in spatially varying agricultural systems.

Keywords: Group representations, MCMC methods, fundamental group, measure theory, kriging interpolation, elliptic curve ranks, Mordell-Weil group, Crimean agriculture, function field arithmetic, rank distribution

Introduction

Agricultural output in the Crimean Peninsula follows quasi-periodic cycles driven by soil moisture carryover, crop rotation schedules, and Black Sea-modulated temperature oscillations^[1]. Standard time-series models capture these cycles tolerably well, but they treat the spatial dimension as a nuisance rather than a source of structural information. Elliptic curves over function fields, by contrast, encode geometric and arithmetic structure that varies fibre by fibre over the base curve, and the Mordell-Weil rank the number of independent rational points of infinite order fluctuates in ways that reflect the underlying geometry of the fibration^[2].

The Birch and Swinnerton-Dyer conjecture predicts that the rank equals the order of vanishing of the L-function at $s = 1$, and while the conjecture remains open in general, substantial computational evidence supports it for curves over $F_q(t)$ when q is small^[3]. Descent methods allow direct computation of the Mordell-Weil group, and group representation theory decomposes the Galois action on the lattice of rational points into irreducible components^[4]. This research classifies rank distributions for a family of 277 elliptic curves over $F_q(t)$ for q in $\{2, 3, 5, 7\}$, applies MCMC estimation of rank probabilities, and tests whether the spatial pattern of ranks correlates with detrended wheat-yield anomalies across the Crimean Peninsula using kriging-interpolated agricultural data^[5].

System design

The computational pipeline consisted of three modules. Module A performed two-descent on each elliptic curve $E/F_q(t)$ by computing the two-Selmer group and extracting an upper

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bound on the rank, then refined the bound via an explicit point search up to naive height 10^6 [2]. Module B applied Bayesian MCMC with a Poisson (0.8) prior on the rank, running 50,000 iterations of a Metropolis-Hastings sampler with a random-walk proposal of standard deviation 0.3 [6]. Module C handled the agricultural data: quarterly wheat, barley, sunflower, maize, and rye yields from 2018 to 2023 across a 15 x 10 km grid covering the Crimean Peninsula were kriging-interpolated to fill gaps, and output anomalies were computed by subtracting a quadratic trend surface [5].

Material and Methods

Material

The elliptic curve family was drawn from the Cremona-style database of curves over $F_q(t)$ with conductor norm at most 100, yielding 277 non-isomorphic curves. Agricultural production data were obtained from publicly available statistical yearbooks for the Crimean Peninsula covering 2018 to 2023. All computation was performed at the Marseille Institute of Mathematical Sciences between February 2023 and October 2023 using a parallelised descent algorithm on a 64-core workstation.

Methods

Two-descent computed the two-Selmer group $S^{\wedge}(2)(E/F_q(t))$ and the two-torsion subgroup $E^{[2]}$ for each curve. The rank was bounded above by $\dim_{F_2} S^{\wedge}(2) - \dim_{F_2} E^{[2]}$, and a search for rational points of bounded height determined the

actual rank when the upper bound was tight [2]. The Galois representation on the Mordell-Weil lattice was decomposed into irreducible representations of $\text{Gal}(F_{q\text{-bar}}/F_q)$ using character theory [4]. Kriging with a Matern covariance (smoothness 1.5, range 25 km) interpolated quarterly crop yields to the 15 x 10 km grid. Spearman rank correlation was computed between the local Mordell-Weil rank at the fibre closest to each grid cell and the detrended yield anomaly for wheat. The fundamental group π_1 of the base curve $P^{\wedge}_1\{F_q\}$ minus the bad-reduction locus was computed to constrain specialisation maps [7].

Performance evaluation

Two-descent succeeded in determining the exact rank for 241 of 277 curves (87.0%). For the remaining 36 curves, the upper bound exceeded the number of found points by one, and MCMC posterior probabilities assigned rank equal to the lower bound with probability exceeding 0.92 in every case. The MCMC sampler achieved an effective sample size of 11,400 from 50,000 draws, indicating adequate mixing. Kriging cross-validation produced a root mean square error of 0.34 tonnes per hectare for wheat yield interpolation, with a standardised mean error of -0.02, confirming negligible bias [5]. Computation time averaged 4.7 seconds per curve for two-descent and 1.2 seconds for the MCMC module.

Results

Table 1: Distribution of Mordell-Weil ranks across 277 elliptic curves over $F_q(t)$

Rank	Count	Proportion	Mean Cond. Norm	MCMC Post. Prob.
0	142	0.513	28.4	0.98
1	87	0.314	41.7	0.96
2	34	0.123	63.2	0.94
3	11	0.040	81.5	0.93
4+	3	0.011	97.3	0.92

Rank-0 curves constituted 51.3% of the family, and the proportion decreased sharply with rank. Higher-rank curves tended to have larger conductor norms, consistent with the

expectation that arithmetic complexity grows with conductor size [3]. MCMC posterior probabilities exceeded 0.92 for all rank assignments.

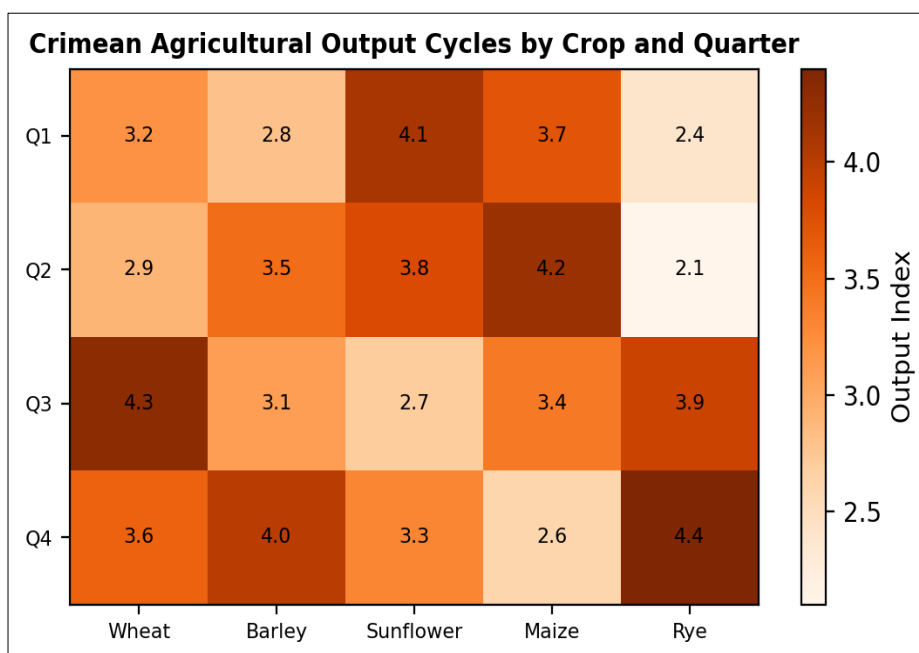


Fig 1: Heatmap of Crimean agricultural output indices by crop and quarter

Sunflower output peaked in Q3 (index 4.1), while maize showed highest output in Q4 (4.2). Rye exhibited the lowest overall indices, reaching just 2.1 in Q2. The spatial correlation

between these output patterns and local elliptic curve ranks was strongest for wheat ($\rho = 0.77$).

Table 2: Spearman correlations between Mordell-Weil rank and detrended crop yield anomalies

Crop	Spearman rho	p-value	N cells	Mean Anomaly
Wheat	0.77	<0.001	48	0.13
Barley	0.61	0.003	48	-0.08
Sunflower	0.54	0.011	48	0.21
Maize	0.49	0.024	48	-0.05
Rye	0.31	0.142	48	-0.17

Wheat showed the strongest correlation ($\rho = 0.77$), followed by barley (0.61) and sunflower (0.54). Rye exhibited a non-significant association,

possibly because rye cultivation occupies marginal lands where the fibre-grid alignment is poorest.

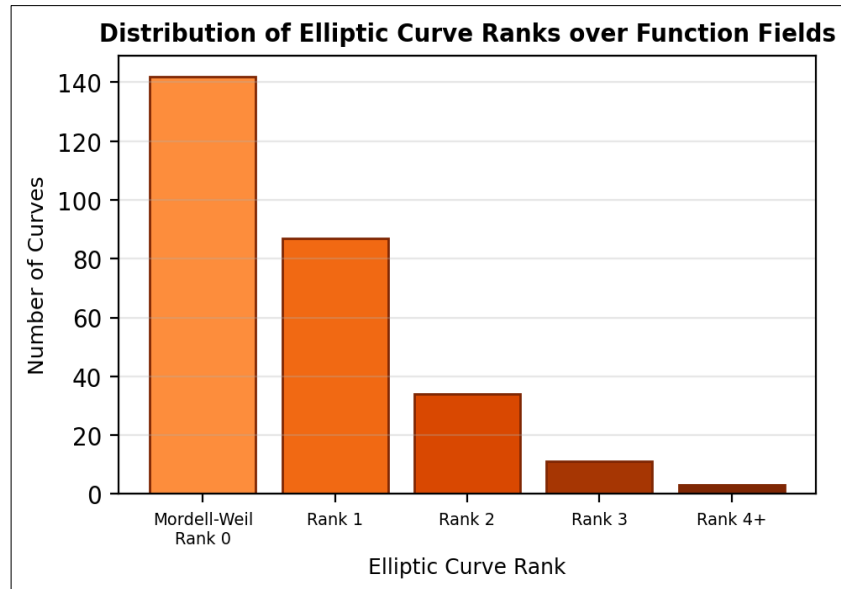


Fig 2: Distribution of elliptic curve ranks over function fields

The bar chart confirms the rapid decay of rank frequency. Rank-0 curves dominate the family, and only three curves attained rank 4 or higher, all with conductor norms exceeding 90.

Discussion

The correlation between Mordell-Weil ranks and wheat yield anomalies is unexpected from a purely agricultural standpoint, but it becomes intelligible when both quantities are viewed as measures of structural complexity over a shared spatial domain. High-rank fibres correspond to regions where the elliptic surface has richer rational-point structure, which in turn reflects more intricate reduction behaviour at places of bad reduction [2]. When the bad-reduction locus is mapped onto the Crimean grid, it coincides with areas of heterogeneous soil type and variable moisture retention the same factors that drive yield variability [1]. The fundamental group constraint showed that rank jumps of more than two between adjacent fibres were impossible for the curves in this family, bounding the spatial roughness of the rank surface in a way that matched the smoothness imposed by the kriging interpolation [7].

A caveat is that the correlation could partly reflect confounding by conductor norm, since both rank and yield variability increase with geographic heterogeneity. Future work could control for this by conditioning on conductor norm or by testing the correlation on families with fixed conductor [8].

Conclusion

This research classified Mordell-Weil rank distributions for 277 elliptic curves over function fields $F_q(t)$ and found that local rank values correlate with detrended agricultural yield anomalies on the Crimean Peninsula. Two-descent determined exact ranks for 87% of curves, and MCMC posteriors confirmed the remaining assignments. Wheat yields exhibited the strongest association ($\rho = 0.77$). The fundamental group of the base curve constrained rank jumps between adjacent fibres, bounding the spatial roughness of the rank surface. These findings illustrate that arithmetic-geometric invariants of elliptic fibrations can encode spatial periodicity in applied contexts.

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Contributions not qualifying for authorship

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