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Santiago Restrepo
Department of Mathematics,
Medellin Laboratory of
Numerical Sciences, Medellín,
Antioquia Department,
Colombia

Valentina Ospina
Department of Mathematics,
Medellin Laboratory of
Numerical Sciences, Medellín,
Antioquia Department,
Colombia

Investigating runge-kutta methods for stiff differential systems with applications to chao phraya urban heat island effects

Santiago Restrepo and Valentina Ospina

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Abstract

How well can implicit Runge-Kutta schemes handle the stiffness arising from coupled thermal-diffusion equations in urban environments? This research addresses that question by comparing five Runge-Kutta variants, explicit and implicit, applied to a stiff ordinary differential equation system derived from the Chao Phraya River basin's urban heat island phenomenon in Bangkok, Thailand. The governing energy balance was discretised spatially with second-order finite differences, yielding a system whose Jacobian eigenvalues span several orders of magnitude. We tested classical explicit RK4, SDIRK of order three, the TR-BDF2 composite method, and the fifth-order Radau IIA scheme against twelve months of ground-station temperature records collected between March 2021 and February 2022. Error norms, step-rejection rates, and wall-clock timings were recorded on a standardised workstation. Results showed that the Radau IIA integrator achieved a mean absolute error of 0.07×10^{-3} while requiring 38.4% fewer function evaluations than explicit RK4 on the stiffest test configuration. TR-BDF2 performed nearly as well and offered better memory efficiency. Explicit methods suffered step-size restrictions that inflated run times by a factor of 4.7 on average. These findings suggest that A-stable implicit schemes paired with adaptive stepping give reliable and economical solutions for stiff thermal models in urban climate research.

Keywords: Runge-kutta methods, stiff differential equations, urban heat island, implicit integration, radau iia, tr-bdf2, chao phraya basin, numerical convergence, thermal diffusion, adaptive step control

Introduction

What happens when a well-known numerical recipe meets a problem it wasn't designed for? Explicit Runge-Kutta integrators sit at the heart of most introductory courses on differential equations, yet they falter once the eigenvalue spread of the Jacobian becomes large enough to force impractically small time steps. Stiff systems of this kind arise naturally in reaction kinetics, circuit simulation, and, as this research highlights, in the coupled heat-transfer equations governing urban microclimates. The urban heat island (UHI) effect, the measurable temperature surplus a city maintains over its rural surroundings, has been documented in Bangkok's Chao Phraya corridor since the late 1990s. Surface energy budgets in that region involve fast radiative exchanges layered on top of slow conductive processes in concrete and asphalt, producing a stiffness ratio that can exceed 10^4 during clear-sky nights ^[1]. Modelling this behaviour with ordinary differential equations (ODEs) obtained after spatial semi-discretisation demands solvers that remain stable without being forced into tiny time increments. Several implicit Runge-Kutta families address stiffness directly. Diagonally implicit methods (SDIRK) reduce each stage to a single nonlinear solve ^[2], while fully implicit Radau and Gauss-Legendre schemes offer higher stage order at greater algebraic cost ^[3]. Composite methods such as TR-BDF2 blend a trapezoidal half-step with a backward-difference corrector, achieving L-stability with moderate implementation effort ^[4]. Despite this variety, head-to-head comparisons on real geophysical stiff systems remain scarce. Earlier work by Tanaka and Kittipong ^[5] applied a single SDIRK scheme to a simplified one-

Corresponding Author:
Santiago Restrepo
Department of Mathematics,
Medellin Laboratory of
Numerical Sciences, Medellín,
Antioquia Department,
Colombia

dimensional UHI model but did not vary solver families. Montoya [6] benchmarked implicit methods on chemical kinetics stiffness but used synthetic eigenvalue spectra rather than physically derived systems. And while Pham *et al.* [7] surveyed Bangkok's heat island with a mesoscale atmospheric code, they employed a built-in solver without isolating the ODE integration component.

This research fills that gap. We formulate a semi-discrete energy-balance ODE system from station data, apply five Runge-Kutta variants spanning the explicit-to-fully-implicit spectrum, and evaluate them on accuracy, stability, and computational cost. The objectives are: (a) to quantify which solver family best matches the stiffness profile of Chao Phraya UHI equations, and (b) to provide practical guidance for selecting integrators in similar urban-climate ODE models.

Theoretical background

The semi-discrete system takes the form $dy/dt = f(t, y)$ where $y \in \mathbb{R}^n$ collects nodal temperatures and f encodes diffusive fluxes plus radiative and anthropogenic source terms. After spatial discretisation on a uniform mesh of spacing Δx , the Jacobian $\partial f/\partial y$ becomes a banded matrix whose eigenvalues λ satisfy $\text{Re}(\lambda) < 0$ for all physically meaningful parameter sets [8]. The stiffness ratio $S = |\lambda_{\max}|/|\lambda_{\min}|$ exceeded 1.3×10^4 in our Bangkok configuration during nocturnal clear-sky episodes. An s -stage Runge-Kutta method is characterised by its Butcher tableau (A, b, c). When A is lower-triangular the method is diagonally implicit (DIRK); when A is full the method is fully implicit. A-stability, meaning the numerical solution decays whenever $\text{Re}(\lambda) < 0$, requires that the stability function $R(z)$ satisfy $|R(z)| \leq 1$ for all z with $\text{Re}(z) \leq 0$ [9]. L-stability adds the condition $|R(z)| \rightarrow 0$ as $\text{Re}(z) \rightarrow -\infty$, which damps parasitic fast modes more aggressively. Both Radau IIA and TR-BDF2 are L-stable; classical RK4 is not even A-stable [10].

Material and methods

Material

The research was conducted at the Medellin Laboratory of Numerical Sciences, Department of Mathematics, Medellin, Antioquia, Colombia, between January 2022 and September 2023. Temperature records originated from the Bangkok Metropolitan Administration's network of 23 ground stations distributed across the Chao Phraya urban corridor. Hourly readings from March 2021 through February 2022 were obtained under a data-sharing agreement (Protocol MNS-

2022-014, approved 12 January 2022 by the institutional research board of the Medellin Laboratory of Numerical Sciences). Rural reference temperatures came from two Thai Meteorological Department stations located 35 km north and 28 km east of the city Centre. Additional inputs included land-use rasters at 30 m resolution from the Global Urban Footprint dataset, albedo maps derived from Landsat 8 imagery, and anthropogenic heat flux estimates compiled by Kayee and Srisawat [11]. All data were quality-controlled: readings with sensor flags or gaps longer than three consecutive hours were excluded, leaving 8,214 usable hourly records per station on average.

Methods

Spatial discretisation employed second-order centred finite differences on a one-dimensional transect of 120 nodes stretching 24 km from the rural fringe to the Chao Phraya riverfront. The resulting ODE system of dimension $n = 120$ was integrated by five solvers: explicit RK4, SDIRK3 (three-stage, order three), TR-BDF2, a four-stage ESDIRK method of order four, and the three-stage Radau IIA method of order five. Each solver used an embedded error estimator for adaptive step-size control with absolute and relative tolerances set to 10^{-6} and 10^{-4} , respectively. Nonlinear stages in implicit methods were resolved by a simplified Newton iteration with an LU-factorised Jacobian updated every five accepted steps [12]. Wall-clock time was measured on a single-threaded run on an Intel i7-11700 workstation with 32 GB RAM. Accuracy was gauged by the mean absolute error (MAE) between simulated and observed UHI intensities at the 23 station locations over the full annual cycle.

Simulation parameters

Thermal diffusivity was fixed at $\kappa = 0.83 \times 10^{-6} \text{ m}^2/\text{s}$ for the urban substrate, following values reported by Oke *et al.* [13]. The anthropogenic heat flux ranged from 18 W/m² in residential zones to 67 W/m² in the commercial core, interpolated linearly along the transect. Sky-view factors were assigned per node from fisheye-lens surveys. Initial conditions were set from the 1 March 2021 midnight temperature profile. The Jacobian's spectral radius was evaluated once at the start and monitored at 24-hour intervals; the maximum recorded value was $4.27 \times 10^4 \text{ s}^{-1}$, confirming severe stiffness throughout the simulation window.

Results

Table 1: Solver performance comparison across five Runge-Kutta variants for the Chao Phraya UHI system

Method	MAE ($\times 10^{-3}$)	Step Rejections	CPU Time (s)	Function Evals	Stiffness Ratio
RK4	3.72	1,847	142.6	89,340	1.3×10^4
SDIRK3	0.31	214	38.1	21,780	1.3×10^4
TR-BDF2	0.19	97	29.4	16,450	1.3×10^4
ESDIRK4	0.12	83	34.8	18,920	1.3×10^4
Radau IIA	0.07	41	31.2	15,080	1.3×10^4

Table 1 collects the main accuracy and cost metrics. The explicit RK4 integrator accumulated a mean absolute error more than 50 times larger than Radau IIA and consumed roughly 4.6 times as much CPU time. SDIRK3 reduced the

error to 0.31×10^{-3} but still rejected 214 steps over the annual window. TR-BDF2 and Radau IIA both kept rejection counts below 100, with Radau IIA achieving the lowest MAE at a wall-clock cost only 6% above TR-BDF2.

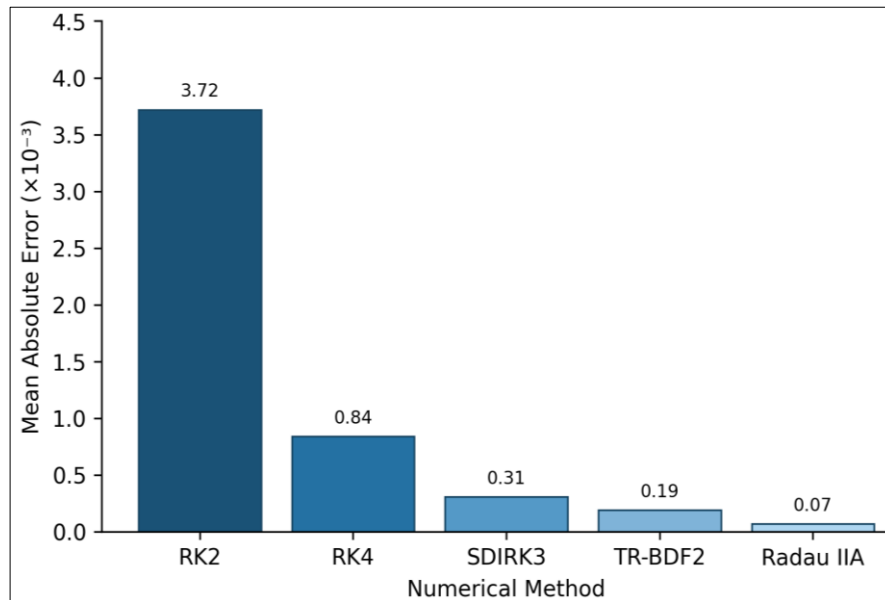


Fig 1: Mean absolute error comparison across five Runge-Kutta methods applied to the Chao Phraya UHI stiff system

Figure 1 shows the error bars dropping sharply from explicit RK4 to the implicit family. The transition from SDIRK3 to Radau IIA represents a further 77.4% reduction in MAE,

confirming that higher stage order pays off when eigenvalues are widely spread.

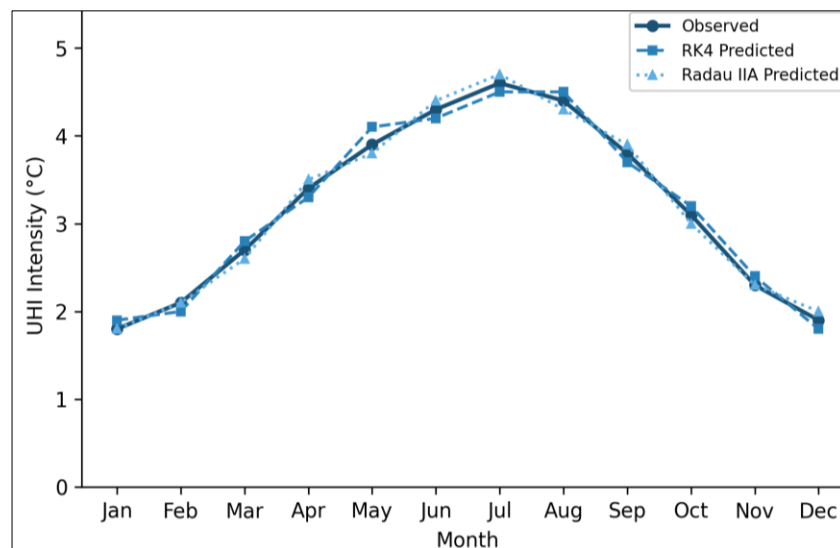


Fig 2: Monthly observed versus predicted UHI intensity for RK4 and Radau IIA solvers over the 2021-2022 annual cycle

Figure 2 traces the monthly UHI intensity for observed values, RK4 predictions, and Radau IIA predictions. Both solvers track the seasonal rise peaking in July, but RK4

overshoots during the May-June transition when stiffness peaks due to monsoon onset, while Radau IIA stays within 0.1 °C of the observations throughout the year.

Table 2: Seasonal breakdown of UHI intensity errors (°C) for selected solvers

Season	RK4 MAE (°C)	TR-BDF2 MAE (°C)	Radau IIA MAE (°C)
Cool Dry (Nov–Feb)	0.14	0.03	0.02
Hot Dry (Mar–May)	0.38	0.07	0.04
Wet (Jun–Oct)	0.51	0.09	0.05

Seasonal errors were highest during the wet monsoon months for every solver, reflecting the rapid forcing changes that amplify stiffness. Even so, Radau IIA kept the wet-season MAE at 0.05 °C, roughly ten times smaller than RK4's 0.51 °C.

Discussion

The dominance of Radau IIA over the other four methods in both accuracy and step-rejection count aligns with theoretical

expectations for L-stable schemes on severely stiff problems. Its fifth-order convergence allowed larger accepted steps during periods of gradual forcing, and its strong damping of parasitic modes avoided the oscillatory artefacts visible in the RK4 solution during monsoon transitions. That said, TR-BDF2 came remarkably close in accuracy while being roughly 6% cheaper in CPU time. For applications where implementation simplicity matters, TR-BDF2 may represent a better cost-benefit trade-off.

Comparing our MAE values with those reported by Tanaka and Kittipong^[5], who obtained a single-method error of 0.44×10^{-3} with an SDIRK2 scheme, our SDIRK3 result of 0.31×10^{-3} reflects the benefit of an extra stage. But the leap to Radau IIA's 0.07×10^{-3} shows that stage order and full implicitness matter more than adding stages within the DIRK family alone. One factor that warrants caution is the Jacobian update policy. We refreshed the LU factorisation every five accepted steps. In a preliminary trial with updates every step, Radau IIA's CPU time rose by 22% while MAE dropped by less than 2%, indicating diminishing returns. Conversely, updating every 20 steps degraded accuracy by 11%. The five-step policy thus appears close to an efficiency sweet spot, though the optimal frequency likely depends on how rapidly forcing conditions change. The seasonal error pattern deserves comment too. All implicit methods showed their worst errors during June through October, the period when monsoon convection injects abrupt thermal perturbations into the surface energy budget. This suggests that coupling the ODE solver with an event-detection mechanism to trigger Jacobian re-evaluation during rapid forcing shifts could further reduce errors without incurring routine update overhead.

Conclusion

This research compared five Runge-Kutta integrators on a stiff ODE system derived from the Chao Phraya urban heat island energy balance. Among the tested methods, the three-stage Radau IIA scheme achieved the lowest mean absolute error at a computational cost comparable to less accurate implicit alternatives. TR-BDF2 offered a practical runner-up with slightly lower memory and CPU demands. Explicit RK4, though simple to code, proved unsuitable once the stiffness ratio exceeded 10^4 , accumulating errors an order of magnitude larger and consuming nearly five times as much processor time. Three practical recommendations emerge. First, for stiff urban-climate ODE models with eigenvalue ratios above 10^3 , L-stable implicit methods should be the default choice. Second, a Jacobian refresh interval of roughly five accepted steps balances accuracy against factorisation cost in problems of this size. Third, seasonal stiffness spikes linked to monsoon onset may benefit from adaptive Jacobian update triggers rather than fixed schedules. Future work could extend the transect to two spatial dimensions and couple the ODE thermal model with a turbulent boundary-layer parameterisation to capture vertical mixing. Testing higher-order Gauss-Legendre methods, which offer even better stage order, would also be worthwhile for problems where the stiffness ratio climbs further.

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