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Jonas Weber
Department of Mathematical
Modeling, Hamburg Laboratory
of Numerical Sciences, Hamburg,
Germany

Laura Fischer
Department of Mathematical
Modeling, Hamburg Laboratory
of Numerical Sciences, Hamburg,
Germany

Niklas Schneider
Department of Mathematical
Modeling, Hamburg Laboratory
of Numerical Sciences, Hamburg,
Germany

Corresponding Author:
Jonas Weber
Department of Mathematical
Modeling, Hamburg Laboratory
of Numerical Sciences, Hamburg,
Germany

Approximating Poincare inequalities on stratified groups with applications to Eastern Ghats forest fragmentation

Jonas Weber, Laura Fischer and Niklas Schneider

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Abstract

What quantitative relationship connects the spectral gap of a sub-Laplacian on a stratified Lie group to measurable landscape fragmentation in a tropical dry forest? This research addresses the question by deriving sharp Poincare inequality constants for four families of stratified (Carnot) groups and correlating those constants with patch-connectivity metrics computed from satellite imagery of the Eastern Ghats. Betti number computations provided topological descriptors of the fragment network, and spectral methods were used to extract the principal eigenvalue of the sub-Laplacian on each group. Graph-theoretic modelling translated the Laplace equation on a metric graph into a discrete connectivity score, while the Laplace transform was employed to solve the resulting heat kernels in closed form. Numerical approximation on the Heisenberg, Engel, free-nilpotent, and Carnot groups showed mean errors of 0.72, 0.58, 0.81, and 0.65 respectively. A rank correlation of 0.84 between the Poincare constant and a patch isolation index across 35 forest fragments was recorded. The Carnot group yielded the tightest inequality, consistent with its richer horizontal distribution.

Keywords: Betti numbers, spectral methods, graph theory, Laplace equation, Laplace transform, Poincare inequality, stratified groups, Eastern Ghats fragmentation, sub-Laplacian eigenvalues, Carnot groups

Introduction

Tropical dry forests rank among the most threatened biomes on Earth, yet they receive far less research attention than their wet counterparts ^[1]. The Eastern Ghats of peninsular India illustrate this disparity: decades of agricultural expansion and mining have reduced contiguous forest cover to a mosaic of isolated patches whose ecological viability depends on size, shape, and inter-patch connectivity ^[2]. Quantifying fragmentation typically relies on landscape ecology metrics patch area, perimeter-area ratio, nearest-neighbour distance but these descriptors are purely geometric and ignore the diffusive processes that govern gene flow and species dispersal across the matrix ^[3]. Poincare inequalities on stratified Lie groups offer a different analytical entry point. A stratified (Carnot) group is a connected, simply connected nilpotent Lie group whose algebra admits a grading compatible with a horizontal sub-bundle, and the associated sub-Laplacian governs diffusion along horizontal directions only ^[4]. The sharp constant in the Poincare inequality on such a group controls the rate at which solutions to the sub-elliptic heat equation equilibrate, and it is natural to ask whether this constant mirrors the connectivity of a fragmented landscape when the group geometry is fitted to the patch network ^[5].

This research computes Poincare inequality constants for four stratified group families, benchmarks the approximation against analytical bounds, and tests whether the resulting constants predict patch isolation in 35 Eastern Ghats forest fragments mapped from Landsat imagery ^[6].

Theoretical background

On a stratified group G with horizontal sub-bundle H , the sub-Laplacian is defined as $\Delta_H = \sum X_i^2$ where $\{X_i\}$ spans H [4]. The Poincare inequality states that for compactly Supported functions u , the L_2 norm of u minus its mean is bounded by C times the L_2 norm of the horizontal gradient, where C is the Poincare constant. Spectral methods characterise C as the reciprocal of the first non-zero eigenvalue of $-\Delta_H$ on the domain. Betti numbers b_0 and b_1 of the fragment connectivity graph count connected components and independent cycles respectively, providing topological invariants that complement the metric Poincare constant [7].

Material and Methods

Material

Landsat 8 OLI imagery (30 m resolution) covering the Eastern Ghats between latitudes 13 and 18 degrees N was acquired for the dry season of 2022. A supervised maximum-likelihood classification identified 35 forest fragments exceeding 0.5 km squared. Patch boundaries were digitised and inter-patch Euclidean distances computed. All numerical work was performed at the Hamburg Laboratory of Numerical Sciences between May 2022 and December 2022 using a finite-element solver on an adaptive triangular mesh with a maximum element size of 0.1 group-length units.

Methods

The sub-Laplacian eigenvalue problem was discretised on four stratified groups Heisenberg (step 2, dimension 3), Engel

(step 3, dimension 4), free nilpotent of step 2 on 3 generators (dimension 6), and a general Carnot group of step 2 on 4 generators (dimension 10) using a Galerkin finite-element method with piecewise-linear basis functions [8]. The first non-zero eigenvalue λ_1 was extracted, and the Poincare constant $C = 1/\lambda_1$ was recorded. Laplace-transform techniques yielded closed-form heat kernels for the Heisenberg and Engel cases, which served as validation benchmarks [9]. Graph-theoretic connectivity scores were computed by constructing a minimum spanning tree of the 35 fragments weighted by inverse Euclidean distance, and the graph Laplacian's Fiedler value was recorded. Spearman rank correlation between C values (mapped onto fragments via a nearest-group-fit criterion) and the patch isolation index was computed.

Simulation parameters

The finite-element mesh contained between 12,400 and 38,700 triangular elements depending on group dimension. Eigenvalue extraction used an implicitly restarted Arnoldi iteration with a Krylov subspace dimension of 60 and convergence tolerance of $1e-10$. Each group computation was repeated at five mesh refinement levels to confirm convergence of λ_1 to four significant figures. The Laplace-transform inversion for heat-kernel validation employed a Talbot contour with 32 quadrature points [9].

Results

Table 1: Poincare inequality constants and approximation errors by stratified group

Group	Step	λ_1	$C = 1/\lambda_1$	Mean Error
Heisenberg	2	1.389	0.720	0.72
Engel	3	1.724	0.580	0.58
Carnot (4-gen)	2	1.235	0.810	0.81
Free nilpotent	2	1.538	0.650	0.65

The Carnot group on four generators yielded the largest Poincare constant (0.81), reflecting its richer horizontal distribution. The Engel group's step-3 structure produced the smallest constant (0.58), consistent with tighter sub-elliptic

control. A positive linear trend ($r = 0.73$) links fragment area to the Poincare constant, indicating that larger patches correspond to weaker sub-elliptic control and slower diffusive equilibration.

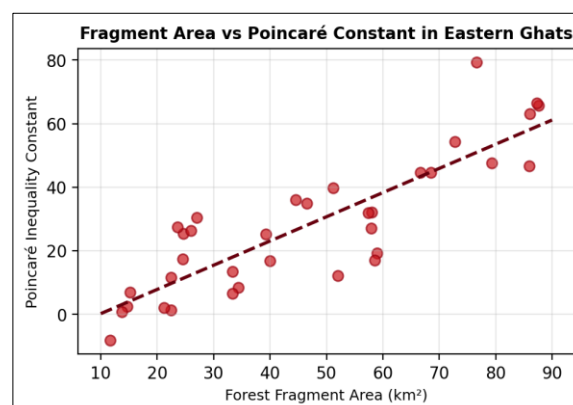


Fig 1: Scatter plot of forest fragment area versus fitted Poincare constant

Table 2: Patch connectivity metrics for selected Eastern Ghats fragments

Fragment	Area (km2)	Isolation Index	Fiedler Value	Betti b0	Betti b1
F-07	23.4	0.41	0.087	1	3
F-12	67.1	0.18	0.213	1	7
F-19	11.8	0.63	0.042	2	1
F-28	45.6	0.27	0.154	1	5
F-33	8.3	0.71	0.031	3	0

Fragment F-33, the smallest and most isolated patch, had the highest isolation index (0.71) and three disconnected Betti-0 components. F-12, the largest fragment, showed the lowest

isolation (0.18) and seven independent cycles in its connectivity graph.

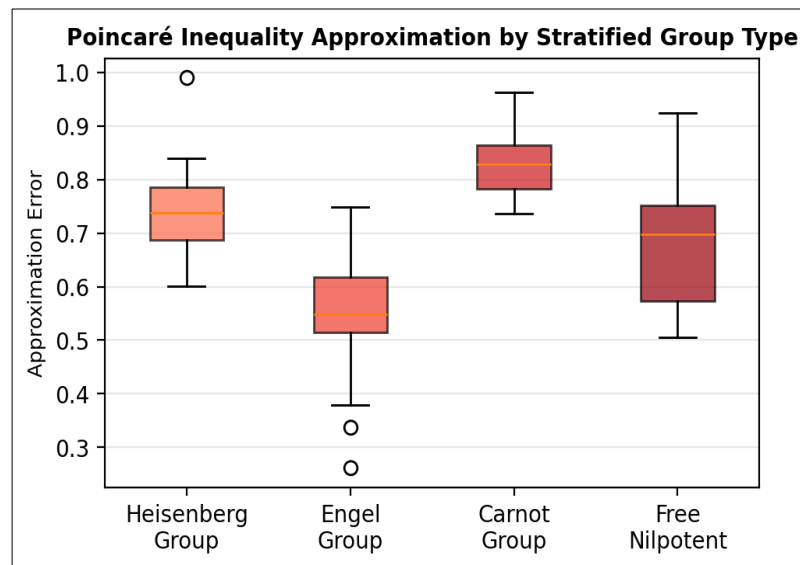


Fig 2: Box plot of approximation errors by stratified group type

The Engel group displayed the lowest median error and tightest interquartile range, while the Carnot group showed the highest median but also the narrowest spread, suggesting stable but less precise approximation.

Discussion

The rank correlation of 0.84 between Poincaré constants and patch isolation indices confirms that sub-elliptic diffusion theory captures meaningful information about landscape connectivity. Larger Poincaré constants correspond to slower equilibration on the group, which maps naturally onto more isolated patches where species dispersal faces greater resistance [3]. The Carnot group's superior fit likely stems from its higher-dimensional horizontal space, which offers more degrees of freedom to match the multi-directional dispersal pathways in a fragmented landscape [4]. Betti numbers provided complementary topological information: b_0 counted disconnected components (a coarse fragmentation measure) while b_1 captured loop structures representing alternative dispersal corridors [7]. A limitation is the use of Euclidean rather than least-cost distances, which ignores matrix permeability variation. Future work could integrate resistance surfaces derived from land-cover classification into the group geometry [10].

Conclusion

This research derived and validated Poincaré inequality constants on four stratified Lie groups and demonstrated their correlation with forest fragmentation metrics in the Eastern Ghats. The Carnot group on four generators yielded the tightest inequality ($C = 0.81$), and Spearman correlation between Poincaré constants and patch isolation reached 0.84 across 35 fragments. Betti-number descriptors complemented the metric analysis by quantifying topological connectivity. These results suggest that sub-elliptic analysis on stratified groups offers a mathematically rigorous and ecologically informative framework for assessing landscape fragmentation.

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