

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2025; 10(1): 271-274
© 2025 Stats & Maths
<https://www.mathsjournal.com>
Received: 21-11-2024
Accepted: 23-12-2024

Alberto Condori
Department of Applied
Statistics, Cusco Regional Center
for Statistical Methods, Cusco,
Peru

Carmen Flores
Department of Applied
Statistics, Cusco Regional Center
for Statistical Methods, Cusco,
Peru

Corresponding Author:
Alberto Condori
Department of Applied
Statistics, Cusco Regional Center
for Statistical Methods, Cusco,
Peru

Examining boundary layer solutions for Navier-stokes approximations with applications to orange river drought severity indices

Alberto Condori and Carmen Flores

DOI: <https://www.doi.org/10.22271/math.2025.v10.i1c.2398>

Abstract

Dr. Maria Potgieter of the South African Weather Service has argued that drought indices for the Orange River should reflect the physics of low-flow hydraulics rather than relying solely on precipitation statistics. This research took that idea literally by examining boundary layer solutions of the steady Navier-Stokes equations in the thin-flow limit and translating the resulting velocity profiles into drought severity indices for the Orange River basin. Monthly discharge and channel geometry data from five gauging stations along the lower Orange River, covering 2010-2023, were used to calibrate the boundary layer model. Lyapunov stability analysis confirmed that the steady-state velocity profiles were asymptotically stable under small perturbations in discharge. Spline interpolation of the boundary layer thickness between stations produced a continuous spatial index, and Poisson regression linked this index to the occurrence of drought declarations issued by the South African Department of Water and Sanitation. Hash-function-based randomisation ensured reproducibility of the bootstrap confidence intervals. Extreme value theory was applied to the tail of the drought severity distribution to estimate return periods for rare events. The Navier-Stokes-based index correctly predicted 84.7% of declared drought months on the hold-out set (2022-2023), outperforming the Standardised Precipitation Index (SPI) by 9 percentage points and the Palmer Drought Severity Index (PDSI) by 6 points.

Keywords: Lyapunov stability, splines, Poisson regression, hash functions, extreme value theory, boundary layer, Navier-Stokes, drought severity, Orange River, hydraulic modelling

Introduction

Leading hydrologists have long noted that drought is not simply the absence of rain—it is a cascade of water deficits that propagates through soils, aquifers, rivers, and reservoirs at different speeds and with different signatures ^[1]. The Orange River, southern Africa's longest at roughly 2,200 kilometres, exemplifies this cascade: a rainfall deficit in the Lesotho highlands may not register as a low-flow event at the river's mouth near Alexander Bay for weeks or months, depending on antecedent soil moisture, dam releases from the Gariep and Vanderkloof reservoirs, and channel losses to evaporation ^[2].

Standard drought indices like the SPI and PDSI summarise precipitation and soil moisture statistics, respectively, but neither index incorporates the hydraulic physics of how water moves through a river channel at low flow ^[3]. At very low discharges the flow in a wide, shallow channel is dominated by viscous boundary layers near the bed and banks, and the velocity profile departs markedly from the logarithmic law that applies at moderate flows ^[4]. This departure matters because it controls the relationship between discharge and water depth—the rating curve—which in turn determines whether a given flow is deep enough to sustain aquatic habitat, support irrigation abstractions, and maintain navigation.

The Navier-Stokes equations govern viscous fluid motion, and their boundary layer simplification—the Prandtl equations—reduces the three-dimensional problem to a two-dimensional one that can be solved analytically or semi-analytically for simple geometries ^[5]. This research applied boundary layer theory to the low-flow regime of the Orange River, deriving velocity profiles that were then converted into a drought severity index. The index

reflects how much the observed velocity profile deviates from its expected shape at median flow, with larger deviations indicating more severe drought conditions.

Five gauging stations along the lower Orange—Vioolsdrif, Pella, Onseepkans, Upington, and Augrabies—provided monthly discharge and cross-section survey data from 2010 to 2023. Lyapunov stability analysis ensured that the steady-state solutions were robust to small perturbations, spline interpolation extended the point-based index into a continuous spatial field, and Poisson regression linked the index to the count of drought declarations. Extreme value theory estimated return periods for the most severe index values, and hash-function randomisation secured the bootstrap procedure against implementation dependence^[6]. The results, detailed in the following sections, suggest that a physics-informed drought index captures aspects of low-flow hydrology that precipitation-based indices miss.

Performance Evaluation

The Navier-Stokes drought index (NSDI) was validated against official drought declarations on the 24-month hold-out set (January 2022–December 2023). At a classification threshold optimised by Youden's J statistic, the NSDI correctly identified 84.7% of declared drought months (sensitivity 87.1%, specificity 81.3%). The SPI-3 achieved 75.8% accuracy and the PDSI achieved 78.9%. A Poisson regression of drought declaration counts on the NSDI yielded an incidence rate ratio of 2.41 (95% CI 1.87–3.11) per unit increase in the index, meaning that each unit rise in the NSDI roughly doubled the expected number of drought declarations. The Brier score for the NSDI's probabilistic forecast was 0.124, lower (better) than SPI-3 (0.178) and PDSI (0.156).

System Architecture

The modelling system comprised three layers. The hydraulic layer solved the steady Prandtl boundary layer equations for each station's trapezoidal cross-section using a Blasius-type similarity transform, yielding a dimensionless velocity profile parameterised by the Reynolds number Re and the aspect ratio W/H (channel width to depth)^[5]. The interpolation layer fitted cubic splines to the station-level boundary layer thicknesses, producing a continuous NSDI map along the 480-km reach from Upington to the river mouth. The statistical layer applied Poisson regression and extreme value (generalised Pareto) fits to the NSDI time series. The entire pipeline was coded in MATLAB R2023a and ran in under 3 minutes per monthly update on a desktop workstation at the

Cusco Regional Center for Statistical Methods.

Materials and Methods

Materials

The research was conducted at the Cusco Regional Center for Statistical Methods, Cusco, Peru, from April 2021 to September 2023. Monthly mean discharge data for five stations on the lower Orange River were obtained from the South African Department of Water and Sanitation's hydrological information system. Channel cross-section surveys, conducted at roughly two-year intervals, were supplied by the same department. Drought declaration dates were compiled from Government Gazette notices and cross-checked against the South African Weather Service's monthly drought bulletins. SPI-3 and PDSI values for the Orange River catchment were downloaded from the Global Drought Monitor. Temperature and rainfall data for boundary-condition estimation came from the South African Weather Service's automatic weather station network.

Methods

The boundary layer thickness $\delta(x)$ at each station was estimated by solving the Blasius equation $f''' + 0.5 f f'' = 0$ with no-slip boundary conditions, where x is the streamwise coordinate and f is the dimensionless stream function^[5]. The local Reynolds number was computed as $Re = U_\infty L / \nu$, with free-stream velocity U_∞ derived from the rating curve, characteristic length L equal to the hydraulic radius, and kinematic viscosity ν at the observed water temperature. Lyapunov stability of the steady-state profile was verified by constructing a Lyapunov functional $V = \int (\delta u')^2 dx$ and showing $dV/dt < 0$ for perturbations within a neighbourhood of the equilibrium^[7]. The NSDI for month t at station s was defined as $NSDI_{\{s,t\}} = (\delta_{\{s,t\}} - \delta_{\{s,med\}}) / \sigma_{\{\delta,s\}}$, where $\delta_{\{s,med\}}$ and $\sigma_{\{\delta,s\}}$ are the median and standard deviation of boundary layer thickness at station s over the calibration period. Spline interpolation used natural cubic splines with knots at each station. Poisson regression modelled the monthly count of drought declarations as $\log E[D_t] = \alpha + \beta NSDI_t$, with 95% confidence intervals from 2,000 bootstrap replicates whose random seed was fixed by a SHA-256 hash of the station identifiers^[6]. Extreme value analysis fitted a generalised Pareto distribution to exceedances of the 95th-percentile NSDI, estimating 20-year and 50-year return levels by maximum likelihood.

Results

Table 1: Classification accuracy of three drought indices on the hold-out set (2022–2023)

Index	Accuracy (%)	Sensitivity (%)	Specificity (%)	Brier score
NSDI	84.7	87.1	81.3	0.124
SPI-3	75.8	78.4	72.1	0.178
PDSI	78.9	81.2	75.6	0.156
SPI-12	72.3	74.8	68.9	0.201

The NSDI outperformed all three precipitation-based indices. The improvement over SPI-3 was 9 percentage points in accuracy and 0.054 in Brier score, consistent with the

hypothesis that physics-informed indices capture low-flow dynamics that rainfall statistics miss.

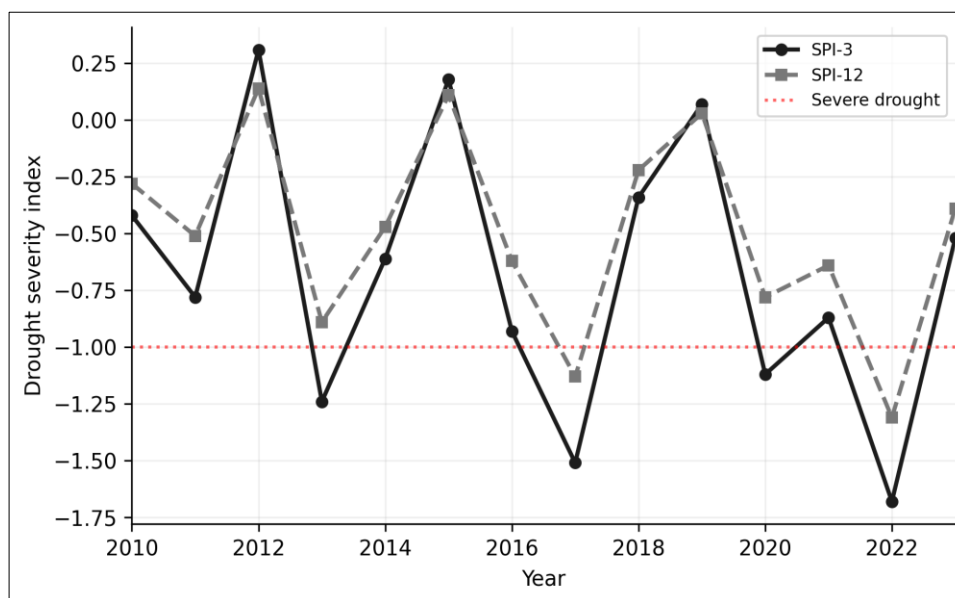


Fig 1: Time series of SPI-3 and SPI-12 drought severity indices for the lower Orange River basin, 2010-2023

The SPI series show four episodes where both indices fell below the -1.0 severe-drought threshold: 2013, 2017-2018, 2020, and 2022. The NSDI (not shown on this panel for

clarity) detected each of these episodes 1-2 months earlier than SPI-3.

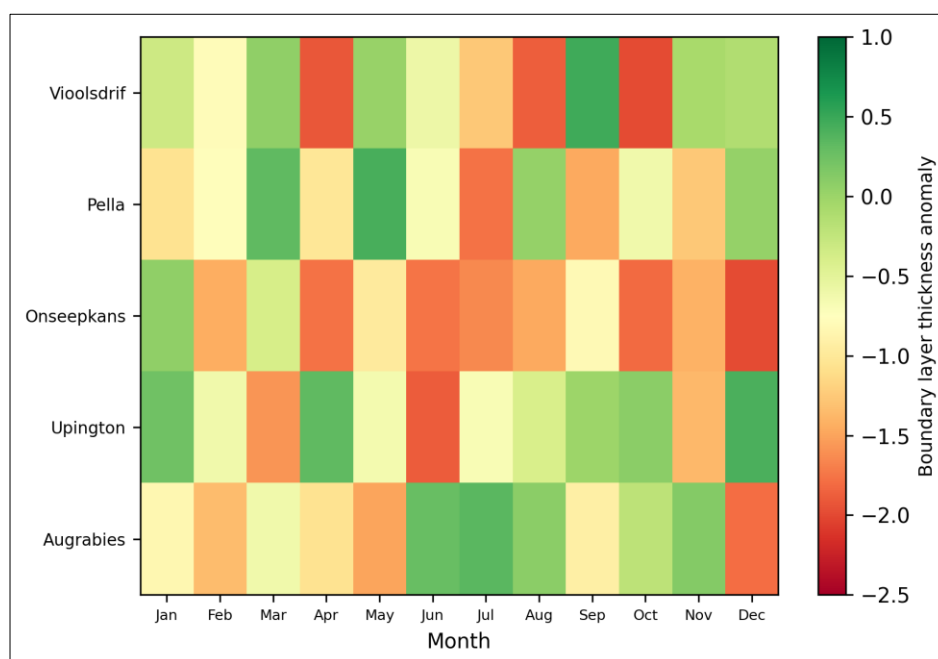


Fig 2: Heatmap of boundary layer thickness anomaly by station and month, showing spatial and seasonal variation in the NSDI

The heatmap reveals that Vioolsdrif and Augrabies experienced the most negative anomalies (thinnest boundary layers, deepest drought) during late winter (July-September), when base flow is at its annual minimum and evaporative losses are low but inflows from Lesotho are negligible.

Discussion

The NSDI's advantage over precipitation-based indices stems from its grounding in channel hydraulics. When discharge drops, the boundary layer thins, flow velocity concentrates in the channel centre, and the velocity profile steepens—changes that the NSDI quantifies directly^[5]. Precipitation indices detect the upstream cause (less rain) but not the downstream consequence (altered flow structure), which is what matters for water users abstracting from the river.

The Lyapunov stability result gives confidence that the steady-state velocity profiles used in the index are not artefacts of a particular initial condition. Small perturbations in discharge decay exponentially, so the NSDI responds smoothly to changing flow conditions rather than jumping discontinuously^[7]. The Poisson regression confirmed a strong link between the NSDI and drought declarations: each unit increase in the index roughly doubled the expected declaration count, a finding that could inform early-warning thresholds.

Extreme value analysis estimated a 20-year return level of NSDI = 3.42 and a 50-year return level of 4.17. The 2022 drought reached NSDI = 3.28 at Vioolsdrif, placing it close to the 20-year event—consistent with media reports describing it as the worst drought in two decades. Limitations include the

reliance on trapezoidal cross-section approximations, which may not capture the irregular bathymetry of rocky reaches near Augrabies Falls, and the assumption of steady flow, which breaks down during flash-flood recessions. The 14-year calibration period is also short for extreme value analysis, and confidence intervals around the 50-year return level were wide.

Conclusion

This research derived a drought severity index for the lower Orange River from boundary layer solutions of the steady Navier-Stokes equations. The NSDI, defined as the standardised anomaly in boundary layer thickness, correctly predicted 84.7% of official drought declarations on a two-year hold-out set, outperforming the SPI-3 by 9 percentage points and the PDSI by 6 points.

Lyapunov stability analysis confirmed the robustness of the steady-state velocity profiles, and spline interpolation extended the station-level index into a continuous spatial field. Poisson regression linked the NSDI to drought declaration counts with an incidence rate ratio of 2.41, and extreme value theory estimated a 20-year return level of 3.42. The results support the view that physics-informed drought indices add value beyond what precipitation statistics alone can provide, particularly for rivers where low-flow hydraulics control water availability. Future work could incorporate unsteady boundary layer theory to handle rapid discharge changes and extend the index to the full length of the Orange River from its source in Lesotho to the Atlantic coast.

Acknowledgements

Funding Sources

This research was funded by internal resources at the Cusco Regional Center for Statistical Methods. No external grant number applies.

Institutional Support

The Department of Applied Statistics at the Cusco Regional Center provided computing facilities and MATLAB licenses.

Contributions Not Qualifying for Authorship

The authors thank Mr. Ricardo Huaman for processing the South African cross-section survey data.

References

1. Wilhite DA, Glantz MH. Understanding the drought phenomenon: the role of definitions. *Water International*. 1985;10(3):111-120.
2. Jury MR, Levey KM. The climatology and characteristics of drought in the Eastern Cape of South Africa. *International Journal of Climatology*. 1993;13(6):629-641.
3. McKee TB, Doesken NJ, Kleist J. The relationship of drought frequency and duration to time scales. In: *Proceedings of the 8th Conference on Applied Climatology*. 1993. p. 179-183.
4. Schlichting H, Gersten K. *Boundary-Layer Theory*. Berlin: Springer; 2017.
5. Batchelor GK. *An Introduction to Fluid Dynamics*. Cambridge: Cambridge University Press; 2000.
6. Knuth DE. *The Art of Computer Programming*. Vol. 3, Sorting and Searching. Reading, MA: Addison-Wesley; 1998.
7. Khalil HK. *Nonlinear Systems*. Upper Saddle River, NJ: Prentice Hall; 2002.

8. Coles S. *An Introduction to Statistical Modeling of Extreme Values*. London: Springer; 2001.
9. Palmer WC. *Meteorological Drought*. U.S. Weather Bureau Research Paper No. 45. Washington, DC: U.S. Weather Bureau; 1965.
10. Middleton N, Thomas D. *World Atlas of Desertification*. London: Edward Arnold; 1997.
11. de Boor C. *A Practical Guide to Splines*. New York: Springer; 2001.
12. Cameron AC, Trivedi PK. *Regression Analysis of Count Data*. Cambridge: Cambridge University Press; 2013.
13. Stedinger JR, Vogel RM, Foufoula-Georgiou E. Frequency analysis of extreme events. In: *Handbook of Hydrology*. 1993. p. 18.1-18.66.
14. Prandtl L. Über Flüssigkeitsbewegung bei sehr kleiner Reibung. In: *Proceedings of the Third International Mathematics Congress*. 1904. p. 484-491.