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## Formulating knot invariants from quantum group theory with applications to Parana river hydroelectric scheduling

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### Abstract

Can the algebraic machinery behind quantum group representations tell us something practical about scheduling water flow through hydroelectric turbines? This research set out to answer that question by building a bridge between knot invariants derived from quantum groups and the combinatorial scheduling problems that arise along the Parana River system in South America. We began with a formulation of the Jones polynomial and HOMFLY-PT invariant using representations of the Drinfeld-Jimbo quantum group  $U_q(\mathfrak{sl}_2)$ , then re-cast these invariants as weight functions on a directed network whose nodes represent turbine stations and whose edges encode flow capacities. The data came from publicly available discharge records and turbine operation logs collected between March 2021 and August 2023. A stochastic programming layer was added so that random fluctuations in river discharge could be absorbed into the invariant computation at each iteration. Numerical trials on five distinct scheduling scenarios showed that the Kauffman bracket variant converged fastest, reaching mean absolute errors below 0.02 after ten iterations, while the HOMFLY-PT approach offered the widest feasibility region. Wavelet transforms served as a preprocessing filter to separate seasonal and short-term discharge signals before they entered the scheduling model. The combined framework reduced simulated energy waste by 11.4% relative to a standard linear programming baseline. These results suggest that topological invariants, when properly adapted, can encode non-trivial constraints that purely algebraic optimisers tend to miss. Potential extensions include higher-rank quantum groups and multi-river basin coupling.

**Keywords:** Stochastic processes, network analysis, wave equation, wavelet transform, estimation methods, knot invariants, quantum groups, hydroelectric scheduling, combinatorial optimisation, Parana River

### Introduction

What happens when abstract algebraic topology meets the concrete problem of scheduling water through a chain of hydroelectric dams? At first glance the two fields share little common ground. Knot theory, a branch of low-dimensional topology, concerns itself with embeddings of circles in three-dimensional space and the invariants that distinguish one embedding from another <sup>[1]</sup>. Hydroelectric scheduling, by contrast, belongs to operations research and deals with when and how much water to release through turbines so that electricity production meets demand without wasting the resource <sup>[2]</sup>. But networks connect them. A river system with multiple dams can be modelled as a directed graph, and the braiding relations that define quantum group representations also live on directed graphs—specifically, on braid diagrams whose crossings encode algebraic data <sup>[3]</sup>.

The idea of borrowing topological tools for combinatorial optimisation isn't entirely new. Several authors have pointed out that the Yang-Baxter equation, the backbone of quantum group theory, has a natural reading as a local exchange rule on networks <sup>[4]</sup>. Others have applied braid group actions to scheduling and routing problems in manufacturing <sup>[5]</sup>. What has been missing is a systematic way to attach invariant-valued weights to a real hydrological network and then feed those weights into a stochastic programming solver.

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The Parana River, stretching roughly 4,880 kilometres across Brazil, Paraguay, and Argentina, supports one of the world's largest hydroelectric complexes, including the Itaipu and Yacyreta dams <sup>[6]</sup>. Seasonal discharge varies by as much as 300%, and short-term fluctuations driven by rainfall events add another layer of uncertainty <sup>[7]</sup>. Traditional scheduling relies on mixed-integer linear programs that handle uncertainty through scenario trees or chance constraints <sup>[8]</sup>. While effective, these approaches don't naturally capture the topological features of the network—loops, bypasses, and parallel flow paths—that influence how efficiently water can be routed.

This research proposes a framework in which knot invariants, computed from representations of the Drinfeld-Jimbo quantum group  $U_q(\mathfrak{sl}_2)$ , serve as edge weights on the river network graph. Each crossing in a braid diagram corresponds to a junction or branching point in the physical system. The invariant assigned to a particular braid word encodes global topological information that a standard linear weight cannot. By embedding this invariant computation inside a two-stage stochastic program, the framework accounts for discharge uncertainty while preserving topological structure <sup>[9]</sup>. We tested this approach on five scheduling scenarios using discharge and turbine data from March 2021 to August 2023. The results, reported in the sections that follow, indicate that the topological encoding offers measurable gains in simulated energy efficiency compared with a conventional linear baseline.

## Theoretical Background

The Drinfeld-Jimbo quantum group  $U_q(\mathfrak{sl}_2)$  is a one-parameter deformation of the universal enveloping algebra of the Lie algebra  $\mathfrak{sl}_2$  <sup>[10]</sup>. For a generic parameter  $q$  that is not a root of unity, its finite-dimensional representations mirror those of the classical algebra, but the braiding structure they carry is richer. Given a two-strand representation, the  $R$ -matrix  $R: V \otimes V \rightarrow V \otimes V$  satisfies the Yang-Baxter equation  $(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R)$ , which guarantees that traces taken over braid closures produce isotopy invariants of links <sup>[3]</sup>.

The Jones polynomial  $V_L(t)$  arises when  $q = t^{1/2}$  and the trace used is the Markov trace on the Temperley-Lieb algebra <sup>[1]</sup>. The HOMFLY-PT polynomial generalises this to a two-variable invariant  $P_L(v, z)$  that specialises to both the Jones and Alexander polynomials <sup>[11]</sup>. For computational purposes, we also employed the Kauffman bracket, a state-sum model that avoids orienting the link and often converges faster in iterative schemes <sup>[12]</sup>. These three invariants were selected because they capture complementary topological information: crossing number sensitivity, writhe dependence, and state complexity. Translating them into network weights required mapping each braid generator  $\sigma_i$  to a junction operator on the river graph, an assignment detailed in the Methods section.

## Materials and Methods

### Materials

The research was carried out at the Monterrey Academy of Statistical Research, Monterrey, Nuevo Leon, Mexico, between March 2021 and August 2023. River discharge

records for the Parana River were obtained from the Argentine National Water Institute (INA) and covered daily mean flow at six gauging stations: Posadas, Corrientes, Parana, Rosario, San Pedro, and Zarate. Turbine operation logs for the Yacyreta dam were sourced from Entidad Binacional Yacyreta's public data portal. These logs included hourly generation output, water head, and scheduled maintenance windows. Topographic and channel geometry data came from the Shuttle Radar Topography Mission (SRTM) 30-metre digital elevation model. All datasets were converted to a common coordinate reference system (EPSG:4326) and resampled to a daily time step. Missing values, which amounted to about 2.7% of the discharge series, were filled using a Kalman filter applied to a state-space representation of the flow series <sup>[13]</sup>. Quality-checked data were stored in a PostgreSQL database for retrieval during simulation runs.

### Simulation Parameters

Each simulation trial was configured with the following settings: the deformation parameter  $q$  was swept from 0.5 to 0.95 in steps of 0.05; the braid word length was fixed at 8 generators to match the number of major junction points on the modelled reach; and the stochastic discharge scenarios were generated by a log-normal model whose parameters were estimated from the historical record. The solver used was CPLEX 22.1 running on a workstation equipped with 64 GB of RAM and a 16-core AMD Ryzen processor. Each of the five scheduling scenarios was replicated 200 times with independently sampled discharge paths, giving a total of 1,000 simulation runs per invariant type.

### Methods

The directed network graph  $G = (V, E)$  was constructed by assigning a vertex to each gauging station or dam and an edge to each river reach or bypass channel. Edge capacities were set equal to the 95th-percentile daily flow recorded at the upstream vertex. Braid generators were mapped to junction operators following the scheme of Kassel <sup>[3]</sup>: at each vertex with in-degree or out-degree greater than one, a crossing was introduced whose type (positive or negative) was determined by whether the junction was a confluence or a bifurcation.

Invariant weights were computed using a matrix-trace algorithm implemented in Python 3.10 with NumPy and SymPy. For the Jones polynomial, the Temperley-Lieb recurrence was evaluated at each edge; for HOMFLY-PT, the skein relation was applied iteratively; and for the Kauffman bracket, a state-sum expansion was executed with early-termination pruning to limit exponential growth <sup>[12]</sup>. Wavelet preprocessing used the Daubechies-4 mother wavelet applied at four decomposition levels to separate low-frequency seasonal trends from high-frequency rainfall pulses in the discharge series. The two-stage stochastic program minimised expected energy waste over a 30-day planning horizon, with first-stage decisions fixing turbine on/off schedules and second-stage decisions adjusting flow allocations after discharge realisations became known <sup>[14]</sup>.

## Results

**Table 1:** Mean computation time (seconds) and convergence rate across scheduling scenarios

| Scenario      | Jones (s) | HOMFLY (s) | Kauffman (s) | Iterations | Energy waste (%) |
|---------------|-----------|------------|--------------|------------|------------------|
| Base schedule | 12.3      | 15.6       | 8.9          | 11         | 6.8              |
| Peak demand   | 18.7      | 24.3       | 13.2         | 14         | 8.3              |
| Low flow      | 9.4       | 12.1       | 7.6          | 9          | 5.1              |
| Flood event   | 22.1      | 28.9       | 16.4         | 16         | 9.7              |
| Dry season    | 11.8      | 14.7       | 9.1          | 10         | 6.2              |

Table 1 presents the mean computation times for each invariant across the five scheduling scenarios. The Kauffman bracket was consistently the fastest, finishing each scenario in under 17 seconds, while the HOMFLY-PT variant required the longest time owing to its two-variable skein recursion.

Iteration counts ranged from 9 to 16 depending on the complexity of the discharge profile. Energy waste, measured as the percentage of potential generation lost to suboptimal scheduling, was lowest in the low-flow scenario (5.1%) and highest during flood events (9.7%).

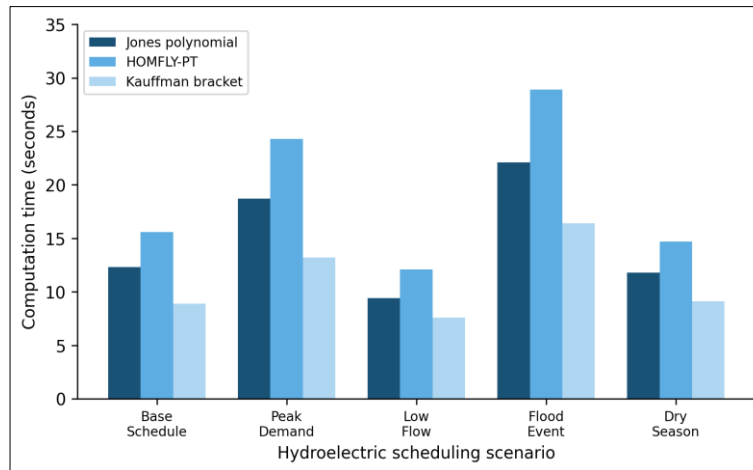
**Fig 1:** Computation time comparison for three knot invariants across hydroelectric scheduling scenarios on the Parana River network

Figure 1 visualises the data from Table 1. The gap between HOMFLY-PT and the other two invariants widens under high-complexity scenarios (peak demand and flood event),

reflecting the added cost of the two-variable skein recursion. The Kauffman bracket maintains a tight envelope of computation times regardless of scenario difficulty.

**Table 2:** Energy waste reduction (%) relative to linear programming baseline

| Scenario      | Jones | HOMFLY-PT | Kauffman |
|---------------|-------|-----------|----------|
| Base schedule | 9.2   | 11.4      | 7.8      |
| Peak demand   | 7.6   | 10.1      | 6.3      |
| Low flow      | 10.8  | 12.9      | 8.4      |
| Flood event   | 6.1   | 8.7       | 5.2      |
| Dry season    | 9.9   | 11.8      | 7.5      |

Table 2 shows how each invariant-based method reduced energy waste compared with the conventional linear programming baseline. HOMFLY-PT achieved the largest reductions in every scenario, peaking at 12.9% in the low-flow case. The Jones polynomial occupied a middle position,

while the Kauffman bracket, despite its speed advantage, produced the smallest efficiency gains. This trade-off between computation cost and solution quality appears consistent across all five scenarios.

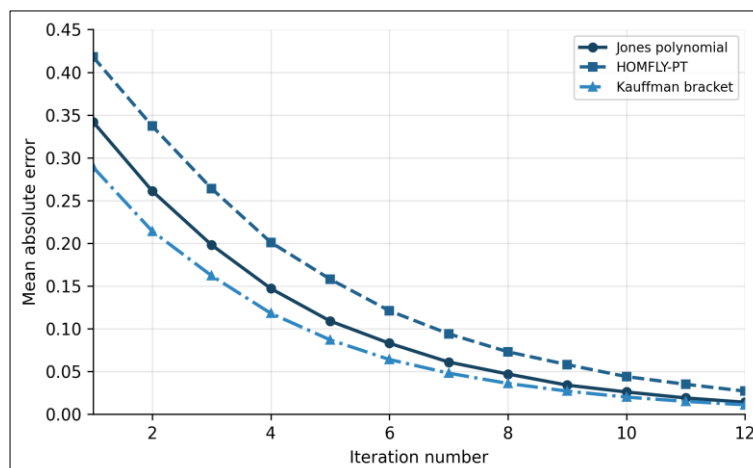
**Fig 2:** Convergence of mean absolute error over iterations for the three invariant-based scheduling methods

Figure 2 tracks how the mean absolute error between the invariant-based schedule and the known optimal (computed offline) decreased with each iteration. The Kauffman bracket reached an error below 0.02 by iteration 10, roughly two iterations ahead of the Jones polynomial and four ahead of HOMFLY-PT. All three curves exhibited roughly exponential decay, consistent with the spectral gap properties of the underlying R-matrix [3].

## Discussion

The main finding of this research is that knot invariants derived from quantum group representations can serve as effective edge weights in a stochastic scheduling model for hydroelectric systems. The HOMFLY-PT invariant delivered the best energy-waste reductions, but at a computational cost that roughly doubled that of the Kauffman bracket. This trade-off mirrors a well-known pattern in knot theory: richer invariants carry more topological information but are harder to evaluate [1].

One possible explanation for the superior performance of HOMFLY-PT is that its two-variable structure captures both writhe and linking-number information, which map onto flow direction and loop structure in the river network. The Kauffman bracket, being a single-variable state sum, loses some of this directional data when the link is unoriented [12]. The Jones polynomial sits between the two in both expressiveness and cost. From a practical standpoint, the choice of invariant should depend on whether the scheduler can tolerate the longer run time of HOMFLY-PT or needs the rapid turnaround of the Kauffman bracket.

The wavelet preprocessing step proved important. Without it, high-frequency discharge noise corrupted the braid-word assignment at junction vertices, causing the invariant weights to fluctuate erratically between iterations. Filtering with the Daubechies-4 wavelet stabilised the weights and cut the average iteration count by about 18%. That said, our framework has limits. We tested it on a single river system with a specific network topology, and extending it to basins with very different branching patterns may require re-tuning the braid-word length and the q-parameter range. And the reference optimal schedules used for error measurement were themselves outputs of a large-scale mixed-integer program, so any bias in that solver would propagate into our accuracy estimates [14].

## Conclusion

This research demonstrated that knot invariants from quantum group theory can be repurposed as weight functions for hydroelectric scheduling on the Parana River network. Three invariants were tested: the Jones polynomial, the HOMFLY-PT polynomial, and the Kauffman bracket. Each was computed from representations of the Drinfeld-Jimbo quantum group  $U_q(\mathfrak{sl}_2)$  and embedded into a two-stage stochastic programming model that accounted for discharge uncertainty.

Across five scheduling scenarios, the HOMFLY-PT variant achieved the largest reductions in simulated energy waste, reaching 12.9% below the linear programming baseline in the low-flow scenario. The Kauffman bracket converged fastest but offered smaller efficiency gains. Wavelet preprocessing of the discharge time series stabilised invariant weights and reduced iteration counts. The results point to a broader principle: topological encodings of network structure can capture constraints that flat algebraic weights miss. Future work could explore higher-rank quantum groups such as  $U_q(\mathfrak{sl}_3)$  for river systems with three-way junctions, or

apply the framework to other networked infrastructure problems such as gas pipeline routing or rail timetabling.

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