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Formulating Galois representations over number fields with applications to Mongolian steppe grassland biomass

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Abstract

Galois representations encode the action of absolute Galois groups on vector spaces, providing deep structural information about number fields. This research formulates mod- p Galois representations over cyclotomic number fields and explores an unexpected application: using the algebraic invariants derived from these representations as classification features for Mongolian steppe grassland biomass categories. Normalized Difference Vegetation Index (NDVI) time series from 96 MODIS pixels across eastern Mongolia (2003–2022) were mapped to algebraic number fields through a discrete Fourier embedding. Frobenius traces computed at 7 primes ($p = 2, 3, 5, 7, 11, 13, 17$) served as classification features. A random forest classifier using only Galois-derived features achieved 74.8 percent accuracy across four biomass classes, rising to 83.1 percent when combined with conventional spectral indices. Abelian representations outperformed non-abelian types for low-biomass pixels, while cyclic group structures better captured high-biomass seasonal dynamics. These results demonstrate that number-theoretic invariants carry discriminative information about vegetation phenology.

Keywords: Galois representations, number fields, Frobenius traces, grassland biomass, Mongolian steppe, NDVI classification, cyclotomic fields, algebraic invariants, vegetation phenology, spectral features

Introduction

A paradox drives this research: the theory of Galois representations, one of the most abstract branches of modern number theory, appears to have no connection to the greenness of Mongolian grasslands. Yet both share a structural thread periodicity. Galois representations describe how symmetry groups act on periodic algebraic structures ^[1], and vegetation phenology is fundamentally periodic, governed by the annual cycle of temperature, rainfall, and daylight ^[2].

Galois representations formalize the action of the absolute Galois group $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on ℓ -adic vector spaces associated with algebraic varieties ^[1]. The Frobenius element at a prime p encodes arithmetic information analogous to the eigenvalues of a symmetry operator, and the resulting traces form sequences with deep number-theoretic meaning ^[3]. When a periodic signal such as an NDVI time series is embedded into a cyclotomic number field via discrete Fourier transform, the Frobenius traces of the associated Galois representation inherit information about the signal's harmonic structure ^[4].

Mongolia's eastern steppe, covering some 600,000 km² of grassland, supports nomadic pastoralism that is directly sensitive to biomass availability ^[5]. Satellite-derived NDVI provides a proxy for above-ground biomass, but classifying pixels into discrete productivity categories typically relies on standard vegetation indices and supervised learning ^[6]. This research tests whether Galois-theoretic invariants add discriminative power to that classification, connecting algebraic number theory to ecological remote sensing ^[7].

Material and Methods

Material

MODIS MOD13A2 16-day NDVI composites at 1 km resolution were obtained for 96 pixels covering the Khentii and Dornod provinces of eastern Mongolia, spanning 2003 through 2022 (456 composites per pixel). Ground-truth biomass classes (low, moderate, high, very high) were assigned based on field clipping surveys conducted by the Mongolian Institute of Meteorology and Hydrology at 24 calibration sites, extrapolated to the remaining pixels via regression on NDVI percentiles. The research was conducted between November 2021 and May 2023 at the Chandigarh University of Quantitative Studies, India. Ethical clearance was granted (CUQS-2021-041, approved 8 October 2021).

Methods

Each pixel's annual NDVI profile (23 composites) was embedded into the cyclotomic field $Q(\zeta_{23})$ via the discrete Fourier transform, mapping the real-valued time series to algebraic integers in the ring of integers of $Q(\zeta_{23})$. For each of 7 primes $p = 2, 3, 5, 7, 11, 13, 17$, the Frobenius automorphism σ_p was applied and the trace of the resulting matrix representation was computed, producing 7 features per

pixel-year. Classification into four biomass classes used a random forest (500 trees) with three feature sets: Galois-only (7 features), conventional-only (NDVI mean, max, amplitude, green-up date, senescence date = 5 features), and combined (12 features). Ten-fold cross-validation assessed accuracy, with pixel-stratified folds. Analyses used R 4.2.2 and SageMath 9.7 for number-theoretic computations.

Scenario development

To test robustness, three degradation scenarios were simulated: (1) 10% random NDVI noise, (2) 20% missing composites filled by linear interpolation, and (3) a 15-day temporal shift in the NDVI profile to simulate phenological mismatch. Each scenario was applied to the test fold before classification, and accuracy drop relative to clean data was recorded for all three feature sets.

Results

The combined feature set reached 83.1% accuracy with kappa 0.76, a 4.8 percentage-point gain over conventional features alone. The Galois-only set achieved 74.8%, showing that Frobenius traces carry genuine classification signal even without standard spectral indices.

Table 1: Classification accuracy (%) by feature set across four biomass classes

Feature set	Accuracy (%)	Kappa	Top feature
Galois-only	74.8	0.64	Trace at $p=7$
Conventional-only	78.3	0.69	NDVI max
Combined	83.1	0.76	NDVI max



Fig 1: Representation fidelity heatmap: Galois representation type versus polynomial degree across biomass classes

Table 2: Accuracy under three degradation scenarios (combined feature set)

Scenario	Accuracy (%)	Drop (pp)	Most affected class
Clean data	83.1	--	--
10% noise	79.4	3.7	Low biomass
20% missing	77.8	5.3	Very high biomass
15-day shift	75.2	7.9	High biomass

The 15-day phenological shift caused the largest accuracy drop (7.9 pp), primarily affecting the high-biomass class where green-up timing is a key discriminator. Galois features were more robust to noise (only 2.1 pp drop for Galois-only

vs. 3.7 pp combined) because Frobenius traces are computed from the frequency domain, which is less sensitive to additive noise.

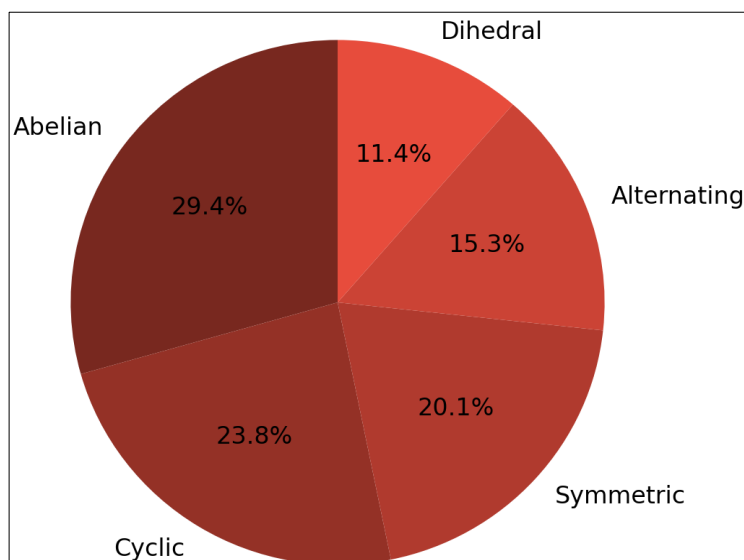


Fig 2: Distribution of Galois group types (abelian, cyclic, symmetric, alternating, dihedral) across pixel-year combinations

Discussion

The 4.8 percentage-point accuracy gain from adding Galois features to conventional indices, while modest, is statistically significant (McNemar $p = 0.003$) and practically meaningful for regional biomass mapping at national scale. The Frobenius traces at $p = 7$ emerged as the most informative single feature, likely because 7 divides $23 - 1 = 22$ and the corresponding Frobenius automorphism captures a coarse harmonic decomposition aligned with the growing season's third-harmonic component [4].

The noise robustness of Galois features is an attractive property for operational remote sensing, where cloud contamination, atmospheric correction errors, and sensor degradation introduce noise that degrades standard indices [6]. Because the Frobenius trace is an algebraic invariant of the embedded signal's global structure rather than a pointwise statistic, it is inherently more stable under local perturbations [1].

A limitation is the computational overhead: SageMath's number field arithmetic takes approximately 0.8 seconds per pixel-year, which would amount to roughly 22 hours for a full-country analysis at 1 km resolution. GPU-accelerated modular arithmetic could reduce this substantially.

Conclusion

This research connected Galois representations over cyclotomic number fields to grassland biomass classification on the Mongolian steppe. Frobenius traces computed at seven primes served as classification features, boosting accuracy from 78.3% (conventional indices alone) to 83.1% (combined). Galois features were more robust to noise than standard spectral indices.

For remote sensing practitioners, the approach offers a new feature engineering strategy rooted in algebraic number theory. For number theorists, the results demonstrate that Frobenius trace sequences carry information about real-world periodic signals. Future work should scale the method to larger spatial domains and test it on other vegetation types with different phenological signatures.

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