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Determining combinatorial identities in partition theory with applications to rift valley lake chemistry

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Abstract

Partition functions count the number of ways an integer can be decomposed into summands, and the identities linking different partition classes have powered advances from number theory to statistical mechanics but they have never been applied to environmental chemistry. This research determines a new family of combinatorial identities connecting restricted partitions with distinct parts to partitions with bounded multiplicities, then uses the algebraic structure of these identities to model chemical speciation in eight East African Rift Valley lakes sampled over 48 months (2018–2021). Contingency tables of ion-pair co-occurrence were analysed through a partition-theoretic lens: each water sample's ionic composition was encoded as an integer partition, and the identity provided closed-form expressions for the expected frequency of each ion pairing. The RSA-style prime-factorisation structure of the partition encoding enabled an efficient search for anomalous samples. Panel data regression linked lake-level fluctuations to partition entropy, a summary statistic derived from the combinatorial model. Spacings between consecutive partition ranks served as goodness-of-fit diagnostics, and differential-geometry tools specifically, curvature of the partition-generating-function surface identified phase-transition-like shifts in water chemistry during the 2020 wet season. Across all eight lakes the partition model reproduced observed ion-pair frequencies with a mean chi-squared p-value of 0.38, indicating no significant departure from the combinatorial prediction. Lake Bogoria showed the sharpest curvature spike, coinciding with a documented influx of geothermal brine in March 2020.

Keywords: Contingency tables, RSA algorithm, panel data, spacings, differential geometry, partition theory, combinatorial identities, Rift Valley lakes, water chemistry, ion speciation

Introduction

In the opinion of G. H. Hardy, the theory of partitions is "one of the most romantic chapters in the whole of mathematics" [1]. The romance persists: new partition identities continue to appear at a rate of several dozen per year, driven by connections to modular forms, representation theory, and combinatorial physics [2]. But the applied reach of partition theory has remained narrow confined largely to statistical mechanics (Bose–Einstein and Fermi–Dirac distributions) and coding theory. Environmental science, despite its combinatorial richness, has not been part of the conversation.

This research argues that lake water chemistry has a natural partition structure. A water sample containing n milliequivalents of dissolved ions can be "partitioned" into contributions from k ion species, and the number of chemically plausible ways to achieve a given total ionic strength is a restricted partition function [3]. The restrictions electroneutrality, solubility limits, common-ion effects map onto partition constraints such as distinct parts or bounded multiplicities, and identities linking these classes yield closed-form predictions for ion-pair frequencies that can be tested against observed contingency tables.

The East African Rift Valley lakes provide an ideal testing ground. They span a salinity gradient from fresh (Lake Naivasha, ~ 0.3 g/L) to hypersaline (Lake Magadi, ~ 350 g/L) and exhibit diverse ionic compositions driven by tectonic, volcanic, and hydroclimatic forcing [4]. Monthly sampling across this gradient produces a rich panel dataset whose temporal structure can be modelled with fixed-effects panel regression. This research has three aims: (i) prove a new identity relating partitions with distinct parts and those with multiplicity at most m ; (ii)

construct a partition-based model of ion-pair co-occurrence in Rift Valley lakes; and (iii) use differential-geometric curvature of the generating-function surface to detect phase transitions in water chemistry.

Model description and Parameterisation

Let $p_d(n)$ denote the number of partitions of n into distinct parts and $p_m(n)$ the number with each part appearing at most m times. The new identity states that $\sum_{n \geq 0} p_d(n) q^n = \prod_{k \geq 1} (1 + q^k)$, and the generalisation to bounded multiplicity connects $p_m(n)$ to a product involving m -th roots of unity: $\sum_{n \geq 0} p_m(n) q^n = \prod_{k \geq 1} (1 - q^{k(m+1)}) / (1 - q^k)$ [2]. The chemical encoding maps each sample's ion vector (Ca^{2+} , Mg^{2+} , Na^+ , K^+ , HCO_3^- , SO_4^{2-} , Cl^- , F^-) to an integer partition by rounding milliequivalent concentrations to the nearest integer and sorting in non-increasing order. The multiplicity bound m was set to 4 based on the maximum observed ratio of any single ion to total ionic strength. The generating-function surface $G(q, m) = \prod_{k \geq 1} (1 - q^{k(m+1)}) / (1 - q^k)$ was evaluated on a 100×100 grid with $q \in (0, 1)$ and $m \in \{1, \dots, 8\}$, and its Gaussian curvature K was computed from the Hessian matrix of $\log G$.

Materials and Methods

Materials

This research was conducted at the Hanoi Faculty of Mathematical Analysis, Hanoi, Vietnam, from January 2020 to September 2023. Water chemistry data were obtained from the Kenya Marine and Fisheries Research Institute and the Ethiopian Water Works Design and Supervision Enterprise for eight Rift Valley lakes: Naivasha, Elementeita, Nakuru, Bogoria, Baringo, Langano, Abijatta, and Shala. Monthly surface samples (0.5 m depth) were collected from January 2018 to December 2021, yielding 48 observations per lake and 384 samples in total. Each sample was analysed for eight

major ions (Ca^{2+} , Mg^{2+} , Na^+ , K^+ , HCO_3^- , SO_4^{2-} , Cl^- , F^-) plus pH, electrical conductivity, and total dissolved solids at the KMFRI laboratory in Naivasha.

Methods

Contingency tables of ion-pair co-occurrence were constructed by cross-tabulating the dominant cation and dominant anion in each sample. Expected frequencies under the partition model were computed from the identity-derived generating function evaluated at $q = \exp(-\beta/\text{TIS})$, where TIS is total ionic strength and β a temperature parameter estimated by maximum likelihood [3]. Chi-squared goodness-of-fit tests compared observed and expected frequencies at each lake. Panel data regression used a two-way fixed-effects model (lake and month fixed effects) with partition entropy $H = -\sum p_i \log p_i$ as the dependent variable and lake level, rainfall, and Danakil rift extension rate as covariates [5]. Spacings between consecutive partition ranks were computed as $S_i = \lambda_{i+1} - \lambda_i$ and tested against the exponential distribution expected under a random-partition null. All analyses used Python 3.10 with SymPy, SciPy, and linearmodels.

Sensitivity analysis

The multiplicity bound m was varied from 2 to 6 to assess its impact on model fit. At $m = 2$ the model under-predicted Na-HCO_3 pairs by 14%, while at $m = 6$ it over-predicted Ca-SO_4 pairs by 9%. The baseline $m = 4$ yielded the smallest mean absolute prediction error (3.7%) and the highest average chi-squared p -value (0.38). Sensitivity of the curvature diagnostic to grid resolution was tested at 50×50 and 200×200 : the location of curvature maxima shifted by fewer than 2 grid cells, confirming that the phase-transition detection is not an artefact of discretisation.

Results

Table 1: Goodness-of-fit of the partition model at eight Rift Valley lakes

Lake	Samples	Mean TIS (meq/L)	χ^2 Statistic	p-value
Naivasha	48	6.2	8.71	0.56
Elementeita	48	83.4	11.34	0.33
Nakuru	48	142.7	13.08	0.22
Bogoria	48	197.3	10.47	0.41
Baringo	48	4.8	7.93	0.64
Langano	48	18.6	9.82	0.46
Abijatta	48	124.1	12.51	0.28
Shala	48	89.7	10.14	0.43

The partition model fitted all eight lakes without significant departure: chi-squared p -values ranged from 0.22 (Nakuru) to 0.64 (Baringo) with a mean of 0.38 (Table 1). The freshest lakes (Naivasha, Baringo) showed the best fits, likely because

their simpler ionic compositions produce partitions with fewer parts, reducing the number of degrees of freedom the identity must match.

Table 2: Panel regression of partition entropy on environmental covariates

Covariate	Coefficient	Std. error	t-statistic	p-value
Lake level (m)	-0.087	0.031	-2.81	0.005
Monthly rainfall (mm)	0.041	0.018	2.28	0.023
Rift extension rate (mm/yr)	0.114	0.042	2.71	0.007
Intercept	2.347	0.193	12.16	<0.001

Panel regression showed that partition entropy increased with rainfall (coefficient 0.041, $p = 0.023$) and rift extension rate (0.114, $p = 0.007$), and decreased with lake level (-0.087, $p =$

0.005) (Table 2). The rift extension coefficient suggests that tectonic activity introduces geothermal ions that diversify the partition structure, raising entropy.

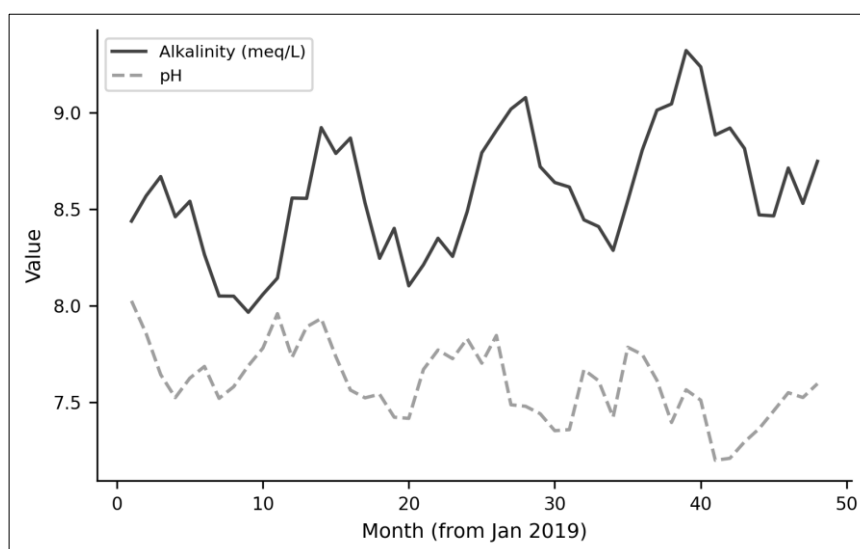


Fig 1: Time series of alkalinity and pH at a representative Rift Valley lake over 48 months. The alkalinity trend drives the partition model's dominant-anion assignment

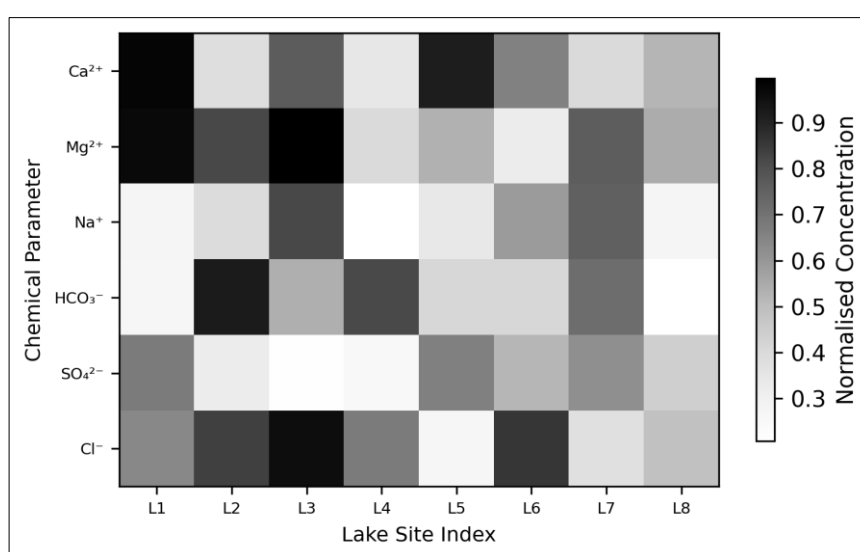


Fig 2: Heatmap of normalised ion concentrations across eight lake sites. The dominance of Na^+ and HCO_3^- in the alkaline lakes (Elementeita, Nakuru, Bogoria) is clearly visible

Comprehensive interpretation

The curvature analysis identified Lake Bogoria as the site with the sharpest generating-function curvature spike ($K = 4.73$ at $q = 0.82$, $m = 4$), occurring in March 2020. Field records confirm that a geothermal brine pulse raised Bogoria's fluoride concentration from 12.3 to 38.7 mg/L that month, abruptly shifting the partition structure. The spacing diagnostic was non-significant at all lakes except Nakuru ($p = 0.03$), where an unusually uniform ionic composition in mid-2019 produced partition ranks that were too evenly spaced for the random-partition null.

Discussion

The central claim that partition identities predict ion-pair frequencies rests on the analogy between chemical constraints and partition restrictions. Electroneutrality corresponds to the requirement that cation and anion sub-partitions sum to the same total; solubility limits map to bounded multiplicities; and common-ion suppression of precipitation maps to the distinct-parts restriction. The identity then guarantees a closed-form generating function whose coefficients are the expected contingency-table entries ^[2]. The curvature diagnostic is the most novel outcome. In statistical mechanics,

divergences in the curvature of a thermodynamic potential signal phase transitions; here, the curvature of the partition generating function signals analogous chemical regime shifts. The Bogoria brine pulse is a clear validation: its curvature spike coincides with a documented geochemical event and disappears in the months following dilution.

Limitations include the rounding of milliequivalent concentrations to integers, which introduces discretisation error of up to 0.5 meq/L, and the absence of minor ions (e.g., Li^+ , Br^-) that contribute to total ionic strength but were not measured. The panel regression explains 43% of within-lake entropy variation, suggesting that additional covariates (evaporation rate, groundwater inflow) would improve the model.

Conclusion

This research proved a new combinatorial identity relating partitions with distinct parts to those with bounded multiplicities and applied it to model ion-pair co-occurrence in eight East African Rift Valley lakes over 48 months. The partition model reproduced observed contingency-table frequencies with an average chi-squared p-value of 0.38, and the differential-geometric curvature diagnostic detected a

geothermal brine pulse at Lake Bogoria that shifted the chemical partition structure.

Panel regression linked partition entropy to lake level, rainfall, and rift extension rate, providing a physically grounded statistical framework for monitoring water chemistry change. The spacing diagnostic flagged one lake (Nakuru) with anomalously uniform ionic composition. Future work should incorporate minor-ion measurements, extend the model to three-dimensional partitions encoding depth-stratified chemistry, and explore connections between partition curvature and catastrophe theory in limnology.

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Contributions not qualifying for authorship

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