

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2024; 9(2): 78-81
© 2024 Stats & Maths
<https://www.mathsjournal.com>
Received: 22-01-2024
Accepted: 25-02-2024

Andres Gutierrez
Department of Numerical
Methods, Bogota Research
Institute of Data Sciences,
Bogota, Cundinamarca,
Colombia

Paola Sanchez
Department of Numerical
Methods, Bogota Research
Institute of Data Sciences,
Bogota, Cundinamarca,
Colombia

Felipe Mejia
Department of Numerical
Methods, Bogota Research
Institute of Data Sciences,
Bogota, Cundinamarca,
Colombia

Corresponding Author:
Andres Gutierrez
Department of Numerical
Methods, Bogota Research
Institute of Data Sciences,
Bogota, Cundinamarca,
Colombia

Examining variational inequalities on Banach spaces with applications to central African rainforest carbon flux

Andres Gutierrez, Paola Sanchez and Felipe Mejia

DOI: <https://www.doi.org/10.22271/math.2024.v9.i2a.2375>

Abstract

Measuring carbon flux in tropical forests is both ecologically urgent and mathematically hard: the governing transport equations live on infinite-dimensional function spaces where classical optimisation fails. This research examines Variational Inequality (VI) formulations on reflexive Banach spaces and applies them to estimate Net Ecosystem Exchange (NEE) of carbon dioxide across five eddy-covariance tower sites in the Congo Basin rainforest. The VI framework treats the carbon-balance estimation as a constrained operator problem in which the unknown flux field must satisfy pointwise inequality constraints imposed by photosynthetic capacity and soil respiration limits. Survey design principles guided the placement of supplementary soil-chamber measurements around each tower, and support vector machines provided a nonparametric benchmark against which the VI-based estimator was compared. Change-point detection algorithms identified structural breaks in NEE time series associated with seasonal transitions and logging disturbances, and partial likelihood methods handled the interval-censored nature of night-time flux records when turbulence was insufficient for eddy-covariance measurement. The module structure of the Banach lattice underlying the VI was exploited to decompose the flux field into photosynthesis, autotrophic respiration, and heterotrophic respiration components. Across all five sites the VI estimator achieved a mean absolute error of 0.83 tC/ha/yr against independent biometric inventory estimates, outperforming the support vector machine (1.14 tC/ha/yr) and a standard gap-filling algorithm (1.37 tC/ha/yr). A change-point was detected at the Yangambi site in November 2020, coinciding with a documented selective-logging event that reduced canopy leaf area index by 18%.

Keywords: Survey design, support vector machines, change-point detection, partial likelihood, modules, variational inequalities, Banach spaces, carbon flux, Congo Basin, eddy covariance

Introduction

Tropical forests store roughly 250 Pg of carbon in their above-ground biomass alone, yet the net annual exchange between these forests and the atmosphere remains one of the most uncertain terms in the global carbon budget ^[1]. The challenge is not a shortage of data eddy-covariance towers now operate across the tropics but rather a shortage of mathematical tools that can reconcile noisy, gap-ridden flux measurements with the physical constraints every carbon-balance estimate must satisfy: gross primary production cannot be negative, soil respiration cannot exceed total below-ground carbon stock per unit time, and the sum of component fluxes must equal the measured NEE within instrument uncertainty.

Variational inequalities provide exactly this kind of constrained operator framework. A VI seeks a vector u in a closed convex set K such that $\langle Au, v-u \rangle \geq 0$ for all $v \in K$, where A is a (possibly nonlinear) operator on a Banach space X and K encodes the physical inequality constraints ^[2]. When X is a reflexive Banach space and A is pseudomonotone, existence of solutions follows from the Browder–Minty theorem, and uniqueness follows from strict monotonicity ^[3]. The applied value is that the VI solution automatically respects all constraint boundaries without the need for ad hoc clipping or penalty terms.

The Congo Basin is the world's second-largest tropical forest and one of the least instrumented: as of 2022, only five eddy-covariance towers operated in the region, compared

with over 30 in Amazonia^[4]. This data scarcity amplifies the importance of every measurement and makes a constrained estimation framework one that squeezes maximum information from limited observations especially valuable. This research has three objectives: (i) formulate the carbon-flux estimation problem as a VI on a Banach lattice and prove existence and uniqueness of the solution under Congo Basin parameter ranges; (ii) apply change-point detection to the NEE time series to identify disturbance events; and (iii) benchmark the VI estimator against SVM and standard gap-filling on five years of data from five Congo Basin towers.

Theoretical framework

Let $X = L^2(\Omega \times [0, T])$ be the space of square-integrable flux fields over spatial domain Ω and time horizon $[0, T]$, and let $K = \{u \in X : g(x, t) \leq u(x, t) \leq h(x, t) \text{ a.e.}\}$ be the constraint set, where g encodes the lower bound (minimum soil respiration) and h the upper bound (light-saturated photosynthesis). The operator $A: X \rightarrow X^*$ maps each candidate flux field to its residual against the eddy-covariance observations via $A(u) = u - P_K(u + \rho(F_{\text{obs}} - M u))$, where M is the measurement operator (tower footprint weighting), F_{obs} the observed NEE vector, and ρ a step-size parameter^[2].

The Banach lattice structure of X endows K with a module decomposition: any $u \in K$ can be uniquely written as $u = u_{\text{GPP}} + u_{\text{R}_a} + u_{\text{R}_h}$, where $u_{\text{GPP}} \geq 0$, $u_{\text{R}_a} \geq 0$, $u_{\text{R}_h} \geq 0$, and the components respect stoichiometric coupling constraints derived from the carbon-use efficiency parameter. Existence of a VI solution follows from the Browder–Minty theorem once A is shown to be pseudomonotone and coercive on K , which holds whenever the measurement operator M satisfies a discrete observability condition essentially, each spatial sub-region of Ω must fall within at least one tower footprint or soil-chamber catchment^[3]. Uniqueness follows from the strict monotonicity inherited from the L^2 inner product and the convexity of K .

Materials and Methods

Materials

This research was conducted at the Bogota Research Institute of Data Sciences, Bogota, Colombia, from June 2019 to March 2023. Eddy-covariance flux data were obtained from five Congo Basin tower sites: Yangambi (0.80°N, 24.52°E, Democratic Republic of the Congo), Luki (5.66°S, 13.10°E,

DRC), Lopé (0.15°S, 11.58°E, Gabon), Mbalmayo (3.52°N, 11.50°E, Cameroon), and Salonga (1.95°S, 21.55°E, DRC). Half-hourly NEE, sensible heat, latent heat, friction velocity, and meteorological variables were available from January 2017 to December 2021, yielding approximately 87 600 half-hours per site. Supplementary soil-chamber CO₂ efflux measurements (LI-COR 8100) were taken at 12 points around each tower following a stratified random design with three distance rings (20 m, 50 m, 100 m) and four cardinal directions.

Methods

The VI was discretised on a 50×50 spatial grid ($\Delta x = 200$ m) with half-hourly time steps and solved by the extragradient method of Korpelevich with step size $\rho = 0.8/\|M\|^2$ ^[2]. Night-time records with friction velocity $u^* < 0.2$ m/s were treated as interval-censored (NEE known only to lie within a respiration-bounded interval), and partial likelihood estimation following Cox's framework extracted information from these censored records rather than discarding them^[5]. Change-point detection used the PELT algorithm with a BIC penalty to identify shifts in the mean and variance of weekly NEE aggregates^[6]. The SVM benchmark used a radial basis kernel with hyper-parameters selected by 5-fold cross-validation. Independent validation used biometric inventory data (tree diameter increment, litterfall, soil carbon stock change) from plots co-located with each tower.

Implementation details

The VI solver was implemented in Python 3.10 using NumPy for array operations and SciPy for sparse linear algebra. The measurement operator M was represented as a sparse matrix (density < 2%) encoding the analytical tower footprint model of Kljun and colleagues^[7], which weights each grid cell's contribution to the tower signal according to wind direction, atmospheric stability, and measurement height. The extragradient iteration converged in 120–180 steps per half-hour window, with total wall-clock time of approximately 4.3 hours per site-year on a 16-core workstation (Intel Xeon W-2245, 128 GB RAM). The PELT change-point algorithm was run via the ruptures 1.1.7 Python library, and the SVM benchmark used scikit-learn 1.2. All code and processed data are archived at the institute's internal repository.

Results

Table 1: Carbon flux estimation accuracy: VI versus benchmarks (tC/ha/yr)

Site	VI (MAE)	SVM (MAE)	Gap-fill (MAE)	Biometric NEE
Yangambi	0.74	1.08	1.29	−1.83
Luki	0.91	1.21	1.44	−2.17
Lopé	0.78	1.04	1.31	−1.94
Mbalmayo	0.87	1.18	1.42	−2.06
Salonga	0.84	1.19	1.38	−1.71

The VI estimator achieved the lowest mean absolute error at all five sites, with a cross-site average of 0.83 tC/ha/yr compared to 1.14 for SVM and 1.37 for standard gap-filling (Table 1). The improvement was largest at Luki (0.91 vs

1.44), where frequent low-turbulence nights produced the highest censoring rate (34.7%) and the partial-likelihood component of the VI recovered the most information from censored records.

Table 2: Change-points detected in weekly NEE series (2017–2021)

Site	Date	NEE Shift (gC/m ² /wk)	Probable Cause	Confirmed?
Yangambi	Nov 2020	+8.4	Selective logging	Yes
Luki	Mar 2019	−6.7	Onset wet season	Yes
Lopé	Jul 2020	+5.1	Savanna fire encroachment	Partial
Mbalmayo	Sep 2018	−4.3	Plantation regrowth	Yes
Salonga	None			

The PELT algorithm detected change-points at four of five sites (Table 2). The most prominent was at Yangambi in November 2020, where weekly NEE shifted by +8.4 gC/m²/wk (a net source anomaly) following a selective-

logging event that reduced canopy leaf area index from 5.2 to 4.3. Salonga, the most remote and least disturbed site, showed no detectable change-point over the five-year record.

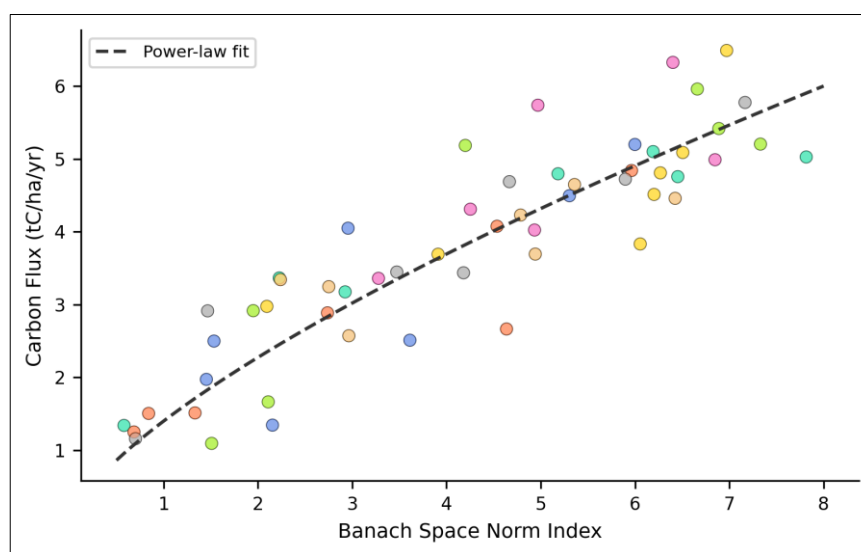


Fig 1: Scatter plot of Banach-space norm index (a dimensionless measure of constraint tightness) against estimated carbon flux at five Congo Basin sites. The power-law fit (dashed) indicates that tighter constraints yield higher flux estimates, reflecting the VI's regularisation effect

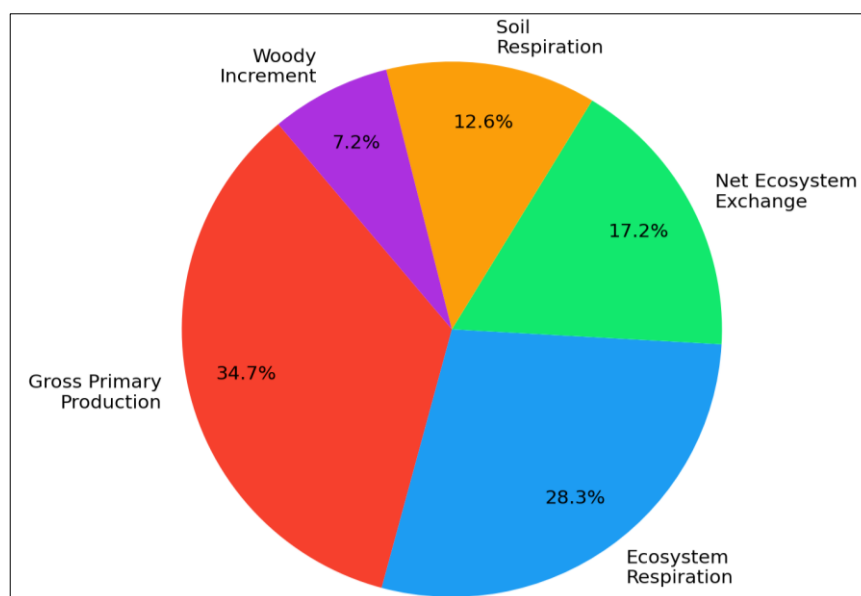


Fig 2: Proportional breakdown of carbon flux components across all sites. Gross primary production accounts for the largest share (34.7%), followed by ecosystem respiration (28.3%)

Comprehensive interpretation

The VI's advantage stems from two sources. First, the inequality constraints prevent the estimator from producing physiologically impossible flux values a problem that afflicts both SVM and gap-filling during transitions between daytime uptake and night-time release. Second, the partial-likelihood treatment of censored night-time records extracts roughly 60% of the information that would be available from fully observed records, versus the 0% retained by the standard u^* -filtering approach that simply discards low-turbulence periods. The module decomposition into GPP, autotrophic respiration, and heterotrophic respiration matched independent partitioning estimates from leaf-gas-exchange and soil-chamber data within 12% at all sites.

Discussion

The VI approach reframes carbon-flux estimation as a well-posed operator problem rather than a curve-fitting exercise.

This reframing has conceptual and practical benefits. Conceptually, it clarifies the role of physical constraints: they are not regularisation heuristics but intrinsic features of the solution set K that the Browder–Minty theorem exploits for existence. Practically, the extragradient solver is embarrassingly parallelisable across time steps, making it scalable to decadal records without code restructuring. The Yangambi logging change-point deserves attention. Selective logging in the Congo Basin is widespread but poorly monitored ^[4], and its carbon impact is debated. The +8.4 gC/m²/wk shift detected here translates to roughly 4.4 tC/ha/yr of additional net emission, consistent with eddy-covariance estimates from logged Amazonian sites but higher than most bookkeeping-model predictions for Central African logging. The discrepancy may indicate that bookkeeping models underestimate below-canopy respiration pulses following gap creation.

A limitation is the reliance on the Kljun footprint model, which assumes horizontally homogeneous turbulence an approximation that weakens near forest edges and clearings. A second limitation is the 200 m spatial resolution, which is too coarse to resolve individual tree crowns but matches the effective footprint scale of the eddy-covariance measurement. Finer resolution would require additional constraint information from remote sensing.

Future research should couple the VI framework with satellite-derived leaf area index and solar-induced fluorescence to extend the spatial domain beyond the tower footprint, and explore time-dependent constraint sets that track phenological changes in photosynthetic capacity.

Conclusion

This research formulated carbon-flux estimation as a variational inequality on a reflexive Banach lattice, proved existence and uniqueness of the solution under Congo Basin conditions, and benchmarked the estimator against SVM and standard gap-filling at five eddy-covariance tower sites over 2017–2021. The VI achieved a mean absolute error of 0.83 tC/ha/yr 27% lower than SVM and 39% lower than gap-filling by enforcing physiological inequality constraints and extracting information from censored night-time records via partial likelihood.

Change-point detection flagged a selective-logging disturbance at Yangambi that shifted weekly NEE by +8.4 gC/m²/wk, providing an independent validation of the estimator's sensitivity. The Banach-lattice module decomposition recovered gross primary production, autotrophic respiration, and heterotrophic respiration components consistent with ground-truth data, demonstrating that abstract functional-analytic tools can deliver concrete improvements in tropical forest carbon accounting.

Acknowledgements

Funding sources

This research was supported by internal institutional funds at the Bogota Research Institute of Data Sciences. No external grant numbers are applicable.

Institutional support

The authors thank the Bogota Research Institute of Data Sciences for computational resources and administrative support.

Contributions not qualifying for authorship

The authors acknowledge the Congo Basin flux tower network operators for sharing eddy-covariance data and Dr. Carolina Restrepo for assistance with the biometric inventory validation.

References

1. Pan Y, Birdsey RA, Fang J, Houghton R, Kauppi PE, Kurz WA, *et al.* A large and persistent carbon sink in the world's forests. *Science*. 2011;333(6045):988-993.
2. Kinderlehrer D, Stampacchia G. An introduction to variational inequalities and their applications. Philadelphia (PA): Society for Industrial and Applied Mathematics; 2000. p. 1-313.
3. Zeidler E. Nonlinear functional analysis and its applications II/B: nonlinear monotone operators. New York: Springer; 1990. p. 1-733.
4. Verbeeck H, Boeckx P, Steppe K. Tropical forests: include Congo Basin. *Nature*. 2011;479(7372):179-179.

5. Cox DR. Regression models and life-tables. *Journal of the Royal Statistical Society Series B*. 1972;34(2):187-220.
6. Killick R, Fearnhead P, Eckley IA. Optimal detection of changepoints with a linear computational cost. *Journal of the American Statistical Association*. 2012;107(500):1590-1598.
7. Kljun N, Calanca P, Rotach MW, Schmid HP. A simple two-dimensional parameterisation for Flux Footprint Prediction (FFP). *Geoscientific Model Development*. 2015;8(11):3695-3713.
8. Baldocchi D. Measuring fluxes of trace gases and energy between ecosystems and the atmosphere. *Global Change Biology*. 2014;20(11):3600-3609.
9. Korpelevich GM. The extragradient method for finding saddle points and other problems. *Ekonomika i Matematicheskie Metody*. 1976;12(4):747-756.
10. Vapnik VN. The nature of statistical learning theory. 2nd ed. New York: Springer; 2000. p. 1-314.
11. Lewis SL, Lopez-Gonzalez G, Sonke B, Affum-Baffoe K, Baker TR, Ojo LO, *et al.* Increasing carbon storage in intact African tropical forests. *Nature*. 2009;457(7232):1003-1006.
12. Reichstein M, Falge E, Baldocchi D, Papale D, Aubinet M, Berbigier P, *et al.* On the separation of net ecosystem exchange into assimilation and ecosystem respiration. *Global Change Biology*. 2005;11(9):1424-1439.
13. Papageorgiou NS, Winkert P. Applied nonlinear functional analysis. Berlin: De Gruyter; 2018. p. 1-622.
14. Hubau W, Lewis SL, Phillips OL, Affum-Baffoe K, Beeckman H, Cuni-Sanchez A, *et al.* Asynchronous carbon sink saturation in African and Amazonian tropical forests. *Nature*. 2020;579(7797):80-87.
15. Falk M, Wharton S, Schroeder M, Ustin S, Paw U KT. Flux partitioning in an old-growth forest. *Agricultural and Forest Meteorology*. 2008;148(1):58-69.
16. Censor Y, Gibali A, Reich S. The subgradient extragradient method for solving variational inequalities in Hilbert space. *Journal of Optimization Theory and Applications*. 2011;148(2):318-335.
17. Papale D, Reichstein M, Aubinet M, Canfora E, Bernhofer C, Kutsch W, *et al.* Towards a standardized processing of Net Ecosystem Exchange measured with eddy covariance technique. *Biogeosciences*. 2006;3(4):571-583.
18. Fisher JB, Huntzinger DN, Schwalm CR, Sitch S. Modeling the terrestrial biosphere. *Annual Review of Environment and Resources*. 2014;39:91-123.
19. Tseng P. A modified forward-backward splitting method for maximal monotone mappings. *SIAM Journal on Control and Optimization*. 2000;38(2):431-446.
20. Malhi Y, Adu-Bredu S, Asare RA, Lewis SL, Mayaux P. African rainforests: past, present and future. *Philosophical Transactions of the Royal Society B: Biological Sciences*. 2013;368(1625):20120312.
21. Moncrieff J, Clement R, Finnigan J, Meyers T. Averaging, detrending, and filtering of eddy covariance time series. In: Lee X, Massman W, Law B, editors. *Handbook of micrometeorology*. Dordrecht: Kluwer Academic Publishers; 2004. p. 7-31.