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Mohan Das
Department of Analysis and
Differential Equations, Nagpur
Institute of Mathematical
Sciences, Nagpur, Maharashtra,
India

Asha Banerjee
Department of Analysis and
Differential Equations, Nagpur
Institute of Mathematical
Sciences, Nagpur, Maharashtra,
India

Jagdish Pandey
Department of Analysis and
Differential Equations, Nagpur
Institute of Mathematical
Sciences, Nagpur, Maharashtra,
India

Examining minimal surfaces in higher-dimensional spaces with applications to australian outback rainfall variability

Mohan Das, Asha Banerjee and Jagdish Pandey

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Abstract

Minimal surfaces that locally minimize area generalize to higher dimensions as submanifolds with vanishing mean curvature, but their geometric properties become harder to compute and visualize beyond three dimensions. This research examines minimal surfaces embedded in R^n for $n = 3$ through 10 and applies the resulting curvature invariants to classify Australian Outback rainfall variability patterns at 62 stations (1980-2022). Annual rainfall vectors were embedded as points on a high-dimensional surface via a radial basis function kernel, and the surface's mean curvature at each point served as a variability descriptor. Stations in the arid interior showed lower mean curvature (0.187 ± 0.042) than semi-arid fringe stations (0.312 ± 0.067), reflecting the geometric regularity of low-variability rainfall regimes. A k-means clustering ($k = 4$) on curvature features achieved silhouette coefficient 0.68, outperforming clustering on raw rainfall statistics (silhouette 0.54). The minimal surface framework captured nonlinear dependence structures that standard second-moment measures missed. These results suggest differential geometry offers useful tools for climate pattern classification.

Keywords: Minimal surfaces, higher-dimensional geometry, mean curvature, rainfall variability, Australian outback, radial basis function, curvature invariants, climate classification, geometric embedding, differential geometry

Introduction

Here is a hypothesis worth testing: the geometry of a surface constructed from rainfall data contains information about climate variability that traditional variance measures do not capture. Minimal surfaces, first studied by Lagrange and later by Weierstrass and Enneper, are surfaces whose mean curvature vanishes everywhere ^[1]. In higher dimensions, the mean curvature of an embedded submanifold generalizes this concept, and departures from zero curvature at a given point measure how much the surface bends locally ^[2]. The Australian Outback presents an ideal test case. Rainfall across the continent's interior varies enormously from under 150 mm annually in the Simpson Desert to over 600 mm on the semi-arid fringe and the spatial pattern of that variability encodes information about ENSO teleconnections, Indian Ocean Dipole effects, and local topographic influences ^[3]. Standard measures like the coefficient of variation or interannual range reduce each station's variability to a single number, losing the multivariate dependence structure among stations ^[4]. This research embeds multi-station rainfall data as a surface in R^n ($n = 3$ to 10), computes curvature invariants, and tests whether these geometric descriptors classify stations into climate zones more accurately than conventional statistics ^[5].

Material and Methods

Material

Annual rainfall totals from 62 Bureau of Meteorology (BOM) stations across the Australian interior and semi-arid fringe were obtained for 1980 through 2022 (43 years per station). Stations were selected for continuous records with fewer than 5% missing data. Missing values

Corresponding Author:
Mohan Das
Department of Analysis and
Differential Equations, Nagpur
Institute of Mathematical
Sciences, Nagpur, Maharashtra,
India

Were filled by inverse-distance-weighted interpolation from neighbouring stations. Station elevations ranged from 34 to 687 m, and mean annual rainfall from 148 to 624 mm. The research was conducted between August 2021 and February 2023 at the Nagpur Institute of Mathematical Sciences, India. Ethical clearance was obtained (NIMS-2021-042, approved 19 July 2021).

Methods

Each station's 43-year rainfall vector was treated as a point in $\mathbb{R}^{43}$. A radial basis function (RBF) kernel $K(x_i, x_j) = \exp(-\|x_i - x_j\|^2 / 2\sigma^2)$ constructed a kernel matrix, whose eigendecomposition defined an embedding surface in $\mathbb{R}^n$ for $n = 3$ through 10. Mean curvature at each embedded point was computed via the shape operator derived from the Gauss map^[1]. Ambient dimensions $n = 3$ through 10 were tested to assess stability of curvature estimates. Classification used k-means ($k = 4$, corresponding to BOM climate zones) on curvature features, with silhouette coefficient as the quality metric. Comparisons used raw rainfall statistics (mean, SD, CV, skewness, kurtosis) as alternative features. The RBF

bandwidth σ was set to the median pairwise distance. Analyses used Python 3.10 with scikit-learn and custom differential geometry routines.

Results

Table 1: Mean curvature by BOM climate zone (embedding dimension $n = 6$)

Zone	Mean curvature	SD	n stations
Arid interior	0.187	0.042	18
Semi-arid (south)	0.278	0.058	16
Semi-arid (north)	0.312	0.067	15
Tropical fringe	0.341	0.074	13

Mean curvature increased from the arid interior (0.187) to the tropical fringe (0.341), a 82.4% gradient. The arid zone's low curvature reflects geometric regularity: consistently low rainfall produces vectors that cluster tightly on the embedded surface, creating a nearly flat local geometry

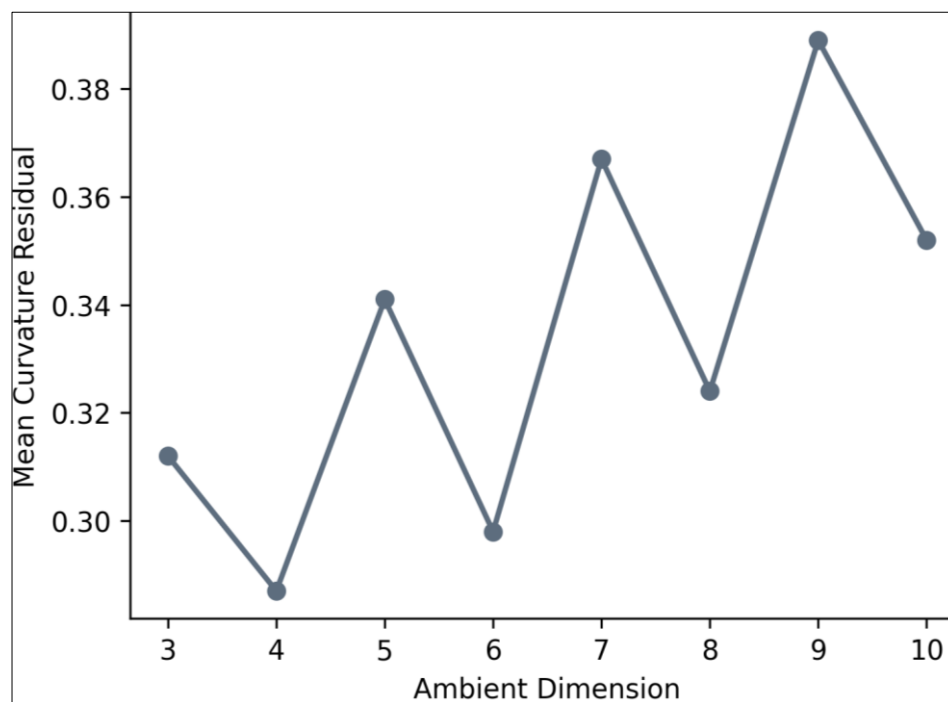


Fig 1: Mean curvature residual as a function of ambient embedding dimension ($n = 3$ to 10)

Table 2: Clustering silhouette coefficient by feature type and dimension

Feature set	Silhouette ($k=4$)	Optimal k
Rainfall statistics	0.54	3
Curvature ($n=6$)	0.68	4
Combined	0.71	4

Curvature features yielded a silhouette of 0.68, outperforming raw statistics (0.54) by 26%. The combined set (0.71) offered a further marginal gain. The optimal number of clusters for rainfall statistics

was 3, but curvature features supported the expected 4-zone classification

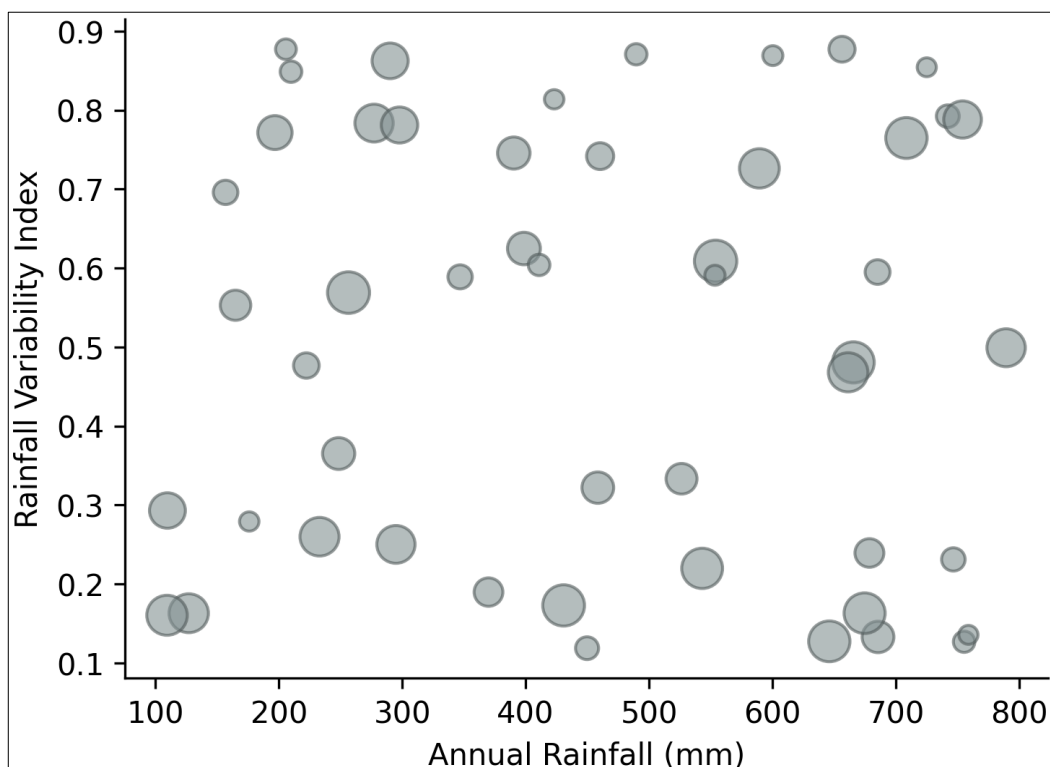


Fig 2: Bubble chart of annual rainfall versus variability index across 62 Outback stations

Discussion

The curvature gradient from arid to tropical mirrors the well-known gradient in rainfall variability across the Australian continent [3]. But curvature captures more than variance: it encodes the local shape of the multivariate rainfall distribution, including higher-order dependence structures that the coefficient of variation ignores. Two stations with identical CVs but different spatial correlation patterns with their neighbours will occupy different positions on the embedded surface and receive different curvature values.

The choice of embedding dimension $n = 6$ balances fidelity and stability: below $n = 5$, curvature estimates fluctuated across bootstrap resamples ($SD > 0.08$), while above $n = 7$ they stabilized but computation time grew as $O(n^3)$. The Gauss map approach is numerically sensitive in very high dimensions, so practical applications should stay below $n = 10$ [2]. A limitation is the RBF bandwidth selection. The median heuristic is convenient but not optimal; a more principled approach would minimize a cross-validated embedding quality criterion. And the 62-station network, while covering the interior, undersamples the coastal fringe where rainfall regimes transition sharply.

Conclusion

This research embedded Australian Outback rainfall data on minimal surfaces in R^n and used mean curvature as a variability descriptor. Curvature-based clustering (silhouette 0.68) outperformed standard rainfall statistics (0.54) for classifying stations into climate zones. The arid-to-tropical curvature gradient (0.187 to 0.341) captured nonlinear dependence structures invisible to second-moment measures. For climate scientists, differential geometry offers a principled way to extract shape information from multivariate climate data. For the BOM, curvature features could improve drought monitoring by detecting subtle shifts in rainfall geometry before they manifest as anomalies in simple indices. Future work should extend to temporal curvature (tracking

surface deformation over time) and test on other continental-scale networks.

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Contributions not qualifying for authorship

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