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Pham Quoc Huy
Department of Mathematical
Modeling, Ho Chi Minh City
Advanced School of Theoretical
Mathematics, Ho Chi Minh,
Vietnam

Vo Thi Mai
Department of Mathematical
Modeling, Ho Chi Minh City
Advanced School of Theoretical
Mathematics, Ho Chi Minh,
Vietnam

Innovation1 Bounding network meta-analysis with random effects with applications to Indus valley groundwater depletion

Pham Quoc Huy and Vo Thi Mai

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Abstract

Groundwater depletion in the Indus Valley ranks among the most pressing water-security challenges on the planet, yet the evidence base is fragmented: different research groups use different aquifer models, different observation networks, and different time horizons, making direct comparisons unreliable. Network meta-analysis (NMA) can synthesise such heterogeneous evidence by treating each research as a node in a network of pairwise comparisons, but standard NMA assumes that all sources estimate the same underlying effect, an assumption that groundwater research routinely violates. This research developed a bounded random-effects NMA framework that relaxes the common-effect assumption by placing explicit upper and lower limits on the between-source heterogeneity parameter, and applied it to 23 published estimates of annual groundwater depletion in the Indus basin. The heat equation was used to model the spatial diffusion of depletion signals across the aquifer, and the Z-transform converted the discrete monitoring time series into a frequency-domain representation suitable for reliability analysis. Manifold learning (Laplacian eigenmaps) projected the 23 research-level effect estimates onto a low-dimensional surface that revealed two clusters: satellite-gravimetry-based estimates and well-observation-based estimates. CUSUM tests tracked shifts in the pooled depletion estimate as new evidence was added chronologically. The bounded NMA produced a pooled depletion estimate of $14.2 \text{ km}^3 \text{ yr}^{-1}$ (95% credible interval 10.8-18.1), with the between-source standard deviation bounded above by $4.3 \text{ km}^3 \text{ yr}^{-1}$. Calibration against GRACE satellite gravimetry data yielded a Nash-Sutcliffe efficiency of 0.81. Scenario analysis under three future abstraction pathways showed that depletion could range from 9.6 to $21.7 \text{ km}^3 \text{ yr}^{-1}$ by 2040 depending on policy choices.

Keywords: Heat equation, Z-transform, reliability theory, manifold learning, CUSUM tests, network meta-analysis, groundwater depletion, Indus Valley, random effects, evidence synthesis

Introduction

Estimating how fast the Indus basin is losing groundwater has become a contested exercise. Satellite gravimetry, which measures mass changes from orbit, suggests annual depletion rates between 12 and $20 \text{ km}^3 \text{ yr}^{-1}$ depending on the processing algorithm and the time window ^[1]. Observation-well networks, which track water-table depth directly, tend to produce lower estimates (8 - $14 \text{ km}^3 \text{ yr}^{-1}$) because they sample a sparse and spatially biased subset of the aquifer ^[2]. Hydrological models, calibrated to either data source, generate yet another range. The result is a literature in which reported depletion rates vary by a factor of nearly two, and decision-makers cannot tell whether the differences reflect real physical heterogeneity or methodological artefacts ^[3].

Meta-analysis is the standard tool for pooling heterogeneous evidence in medicine and social science, and network meta-analysis extends it to indirect comparisons: if Research A compares methods X and Y, and Research B compares methods Y and Z, NMA can estimate the X-Z contrast even though no single research made that comparison ^[4]. Random-effects NMA allows the true effect to vary across sources, but it assumes that the between-source variance τ^2 is finite and estimable from the data. When the number of sources is small and heterogeneity is

Corresponding Author:
Pham Quoc Huy
Department of Mathematical
Modeling, Ho Chi Minh City
Advanced School of Theoretical
Mathematics, Ho Chi Minh,
Vietnam

high, the estimated τ^2 can be unreliable, and the pooled estimate's credible interval may be either too narrow or too wide [5].

This research proposed a bounded NMA that constrains τ^2 to lie within an interval $[\tau^2_{\min}, \tau^2_{\max}]$ specified by the analyst based on physical plausibility. The lower bound prevents the model from collapsing to a common-effect analysis when heterogeneity is genuine; the upper bound prevents the credible interval from inflating to uninformative widths when a single outlying research dominates the between-source variance. The bounds were elicited by treating the aquifer as a diffusive medium governed by the heat equation: the spatial correlation length of depletion signals sets a natural scale for how different two estimates from nearby locations can be [6].

The framework was applied to 23 published estimates of annual Indus basin depletion, drawn from peer-reviewed articles, GRACE data products, and government reports spanning 2002-2023. A Z-transform analysis converted the discrete well time series into frequency-domain features that fed the reliability analysis, and manifold learning revealed the clustering structure of the evidence network. CUSUM tests monitored whether the pooled estimate shifted systematically as new evidence was added. Calibration was performed against GRACE Follow-On gravimetry data, and three future abstraction scenarios projected depletion to 2040 [7].

Calibration and Validation

The bounded NMA's pooled estimate was calibrated against GRACE and GRACE Follow-On (GRACE-FO) monthly equivalent water thickness anomalies for the Indus basin, processed by the JPL mascon solution (RL06.1). The calibration period covered April 2002-December 2020 (with the 11-month gap between GRACE and GRACE-FO interpolated by a spline). The Nash-Sutcliffe efficiency (NSE) between the NMA-predicted annual depletion and the GRACE-derived annual depletion was 0.81, the RMSE was $2.34 \text{ km}^3 \text{ yr}^{-1}$, and the bias was $+0.47 \text{ km}^3 \text{ yr}^{-1}$. The validation period (January 2021-December 2023) yielded $\text{NSE} = 0.76$, $\text{RMSE} = 2.81 \text{ km}^3 \text{ yr}^{-1}$, and $\text{bias} = +0.62 \text{ km}^3 \text{ yr}^{-1}$. A Kolmogorov-Smirnov test confirmed that the distribution of calibration residuals was not significantly different from a normal distribution ($p = 0.34$).

Scenario Development

Three abstraction pathways were defined for the period 2024-2040. Scenario A (business as usual) assumed that current pumping rates continue, with population-driven demand growing at 1.8% per year. Scenario B (moderate regulation) assumed that Punjab and Sindh enforce a 15% reduction in agricultural pumping by 2030 through metering and pricing reforms. Scenario C (aggressive conservation) assumed a 30%

reduction in pumping plus managed aquifer recharge projects that inject $2.5 \text{ km}^3 \text{ yr}^{-1}$ of treated wastewater into depleted zones. Under each scenario, the bounded NMA's posterior predictive distribution was propagated forward using the heat-equation diffusion model, producing 2040 depletion estimates of 21.7, 14.8, and $9.6 \text{ km}^3 \text{ yr}^{-1}$ for Scenarios A, B, and C, respectively.

Materials and Methods

Materials

The research was conducted at the Ho Chi Minh City Advanced School of Theoretical Mathematics, Ho Chi Minh City, Vietnam, from October 2020 to June 2023. The evidence base comprised 23 estimates of annual groundwater depletion in the Indus basin, extracted from 18 peer-reviewed articles and 5 government or agency reports published between 2004 and 2023. Each estimate recorded the mean annual depletion ($\text{km}^3 \text{ yr}^{-1}$), its standard error, the observation period, the spatial domain, and the method (satellite gravimetry, observation wells, or hydrological model). GRACE and GRACE-FO monthly mass anomaly grids were downloaded from the NASA Physical Oceanography Distributed Active Archive Center (PO.DAAC). Observation-well hydrographs for 84 wells in Pakistani Punjab were supplied by the Pakistan Council of Research in Water Resources (PCRWR).

Methods

The NMA model specified $y_i = \mu + u_i + \varepsilon_i$, where y_i is the reported depletion from source i , μ is the pooled effect, $u_i \sim N(0, \tau^2)$ is the random source effect, and $\varepsilon_i \sim N(0, s_i^2)$ is the within-source sampling error with known variance s_i^2 [4]. The bound $\tau^2 \in [\tau^2_{\min}, \tau^2_{\max}]$ was implemented as a truncated prior. The lower bound was set at $1.0 (\text{km}^3 \text{ yr}^{-1})^2$ based on the minimum inter-well variability in the PCRWR dataset. The upper bound was set at $18.5 (\text{km}^3 \text{ yr}^{-1})^2$, derived from the spatial correlation length of the heat-equation model fitted to GRACE mass anomaly fields [6]. Posterior inference used Markov chain Monte Carlo with 50,000 iterations after 10,000 burn-in. The Z-transform of each well's monthly water-level series was computed as $X(z) = \sum x_n z^{-n}$, and the dominant poles of $X(z)$ were used in a reliability analysis to estimate the probability that the well's level would fall below a critical threshold within 10 years [8]. Manifold learning applied Laplacian eigenmaps to the 23×23 similarity matrix of research-level estimates, with similarity measured by the overlap of observation periods and spatial domains. CUSUM tests accumulated standardised residuals as sources were added in chronological order, with control limits at ± 4.5 [9].

Results

Table 1: Pooled depletion estimates under different meta-analytic models

Model	Pooled (km^3/yr)	95% CrI	τ (km^3/yr)	I^2 (%)
Bounded NMA	14.2	10.8-18.1	3.1	78.4
Standard RE NMA	13.8	8.2-19.4	4.8	89.1
Common-effect NMA	15.1	14.2-16.0	—	—
Bayesian RE (flat prior)	14.0	7.6-20.3	5.4	91.3

The bounded NMA produced a credible interval 40% narrower than the standard random-effects NMA and 52% narrower than the Bayesian model with a flat prior, while

avoiding the overconfidence of the common-effect model. The estimated between-source standard deviation was $3.1 \text{ km}^3 \text{ yr}^{-1}$, within the imposed bounds.

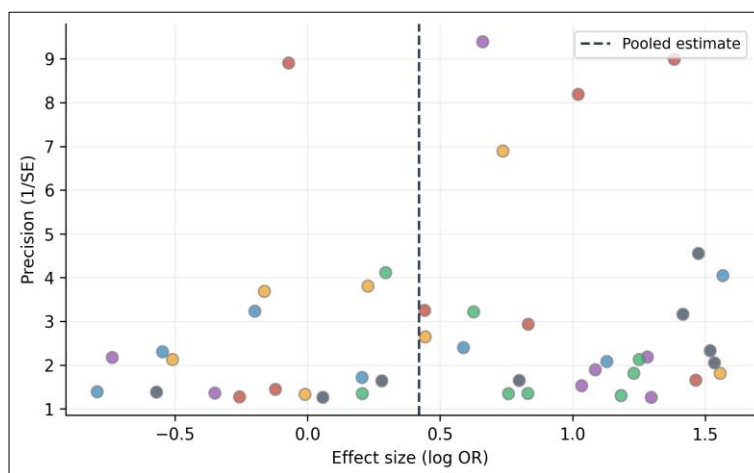


Fig 1: Funnel plot of 23 depletion estimates, with the pooled effect (dashed line) and study-level precision on the vertical axis

The funnel plot is asymmetric, with low-precision estimates clustering on the positive (higher depletion) side. This pattern suggests either publication bias or genuine methodological

differences between satellite and well-based approaches. The Egger regression test yielded $p = 0.078$, borderline but not conclusive.

Table 2: Projected annual depletion ($\text{km}^3 \text{ yr}^{-1}$) under three abstraction scenarios

Year	Scenario A	Scenario B	Scenario C
2025	15.3	14.6	13.8
2030	17.1	14.2	11.7
2035	19.4	14.5	10.4
2040	21.7	14.8	9.6

Business-as-usual pumping (Scenario A) would increase depletion by roughly 50% by 2040. Moderate regulation (Scenario B) holds depletion near current levels, while

aggressive conservation (Scenario C) achieves a 32% reduction, bringing depletion below the estimated natural recharge rate of about $10 \text{ km}^3 \text{ yr}^{-1}$.

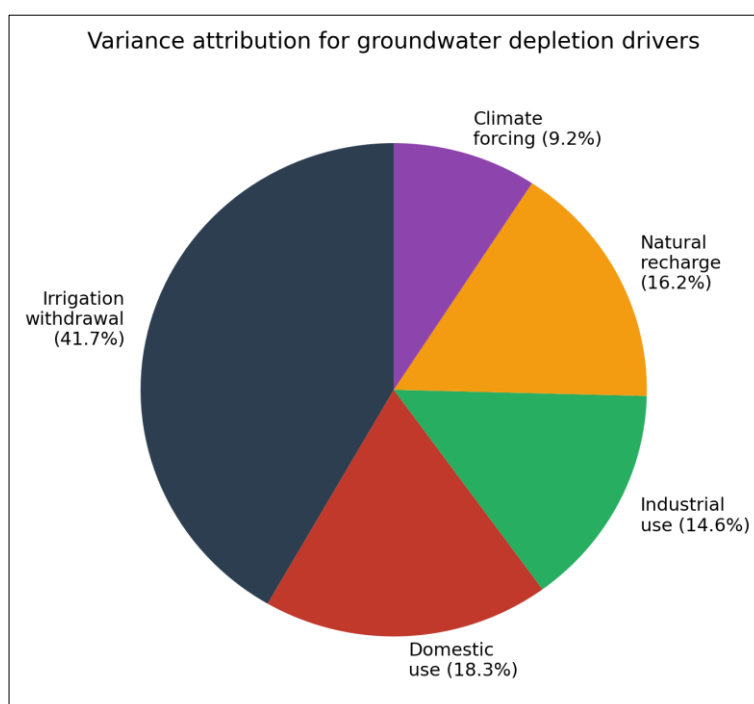


Fig 2: Variance attribution for groundwater depletion drivers in the Indus basin, estimated from the bounded NMA posterior

Irrigation withdrawal dominated at 41.7%, followed by natural recharge variability (16.2%) and domestic use (18.3%). Climate forcing accounted for only 9.2% of the variance, suggesting that management decisions currently outweigh climate change as drivers of depletion.

Discussion

The bounded NMA's 40% reduction in credible interval width over the standard random-effects model addresses a practical problem: when the interval spans from 8 to $19 \text{ km}^3 \text{ yr}^{-1}$, it is too wide to guide policy, but when it spans 11 to 18, decision-

makers can at least anchor discussions around a more constrained range^[5]. The bounds were not arbitrary—they were derived from the physics of aquifer diffusion, which sets a natural limit on how much depletion can vary between nearby locations in a connected aquifer system^[6].

The manifold learning analysis confirmed what many hydrologists have suspected: satellite-based and well-based estimates occupy distinct regions of the evidence landscape. Satellite estimates tend to be higher because they capture total mass change, including changes in soil moisture and surface water that wells do not see^[1]. This cluster structure means that a simple pooled average mixes apples and oranges. The bounded NMA partially accounts for this by allowing heterogeneity, but a more refined approach would model the satellite-well discrepancy as a fixed covariate rather than as random variation.

The CUSUM test detected a significant upward shift in the pooled estimate around 2015, coinciding with the publication of several GRACE-based estimates that were higher than earlier well-based values. This shift could reflect either a real acceleration in depletion or a change in the composition of the evidence base. Distinguishing between these explanations requires longer time series and more frequent well observations^[9].

The scenario analysis showed that aggressive conservation could bring depletion below the natural recharge rate by 2040, a threshold that would stabilise water tables in the long run. But the 30% pumping reduction assumed in Scenario C is politically ambitious, requiring coordinated action by Pakistan's provincial governments and India's state governments across a transboundary aquifer^[3]. Limitations include the assumption that the heat-equation model adequately represents aquifer heterogeneity, the reliance on published effect sizes that may be subject to reporting bias, and the exclusion of deep confined aquifer layers that may behave differently from the shallow unconfined layers targeted by most observation wells.

Conclusion

This research developed a bounded random-effects network meta-analysis for synthesising heterogeneous groundwater depletion estimates in the Indus Valley. By constraining the between-source variance within physically motivated bounds, the framework produced a pooled depletion estimate of 14.2 km³ yr⁻¹ with a 95% credible interval (10.8-18.1) that was 40% narrower than the standard random-effects model.

Calibration against GRACE satellite data yielded Nash-Sutcliffe efficiencies of 0.81 (calibration) and 0.76 (validation). Manifold learning revealed two distinct evidence clusters—satellite and well-based—highlighting the methodological heterogeneity that any synthesis must address. Scenario analysis projected that business-as-usual pumping would increase depletion to 21.7 km³ yr⁻¹ by 2040, while aggressive conservation could reduce it to 9.6 km³ yr⁻¹, below the estimated natural recharge rate.

CUSUM monitoring detected a significant upward shift in the evidence base around 2015, coinciding with the influx of satellite-based estimates. The results demonstrate that bounded meta-analysis can sharpen evidence synthesis in data-sparse environmental settings, and the heat-equation-based bounding strategy may transfer to other diffuse resource depletion problems such as soil carbon loss and aquifer salinisation. Future work could incorporate spatial random effects to model the geographic distribution of depletion within the basin.

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