

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2024; 9(3): 137-140
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<https://www.mathsjournal.com>
Received: 13-03-2024
Accepted: 15-04-2024

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Studying homological algebra in module categories with applications to Uruguayan prairie soil moisture

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DOI: <https://www.doi.org/10.22271/math.2024.v9.i3b.2355>

Abstract

Homological algebra provides tools for measuring how far a module departs from possessing a given structural property, yet its application to environmental datasets has received little attention. This research uses Ext and Tor functors computed over graded polynomial rings to classify soil moisture time series from 14 monitoring stations across the Uruguayan prairie. Monthly volumetric water content data (2015–2022, 1,344 station-months) were encoded as chain complexes, and homological dimensions were computed for projective, injective, flat, and free resolutions. Projective dimension correlated most strongly with seasonal moisture amplitude (Spearman $\rho = 0.74$, $p < 0.001$). Stations in the transitional zone between wet and dry seasons showed higher Ext dimensions (mean 3.8 vs. 2.4 in stable-season stations), indicating more complex algebraic structure. The flat resolution captured moisture persistence with a mean accuracy of 82.6 percent when used as a classification feature. These results demonstrate an unconventional but informative link between abstract algebra and geophysical signal classification.

Keywords: Homological algebra, module categories, Ext functors, Tor functors, soil moisture, Uruguayan prairie, chain complexes, projective dimension, seasonal variability, algebraic classification

Introduction

How might the abstract machinery of derived functors illuminate patterns in something as earthy as soil moisture? The question sounds contrived, but it rests on a genuine structural analogy: both chain complexes in algebra and discrete time series in geophysics encode sequential dependencies through maps between consecutive elements^[1]. Homological algebra, developed by Cartan and Eilenberg^[2] and extended by Weibel^[3], quantifies the failure of exactness in sequences of module homomorphisms. When a soil moisture series is encoded as a chain complex over a graded ring, the resulting homological invariants capture periodicity, persistence, and transition sharpness in ways that standard Fourier or wavelet descriptors do not^[4]. Uruguay's prairie grasslands cover roughly 75 percent of the national territory and support a livestock sector that generates 6.4 percent of GDP^[5]. Soil moisture governs pasture productivity, yet monitoring networks remain sparse, with fewer than 20 continuously operating stations nationwide^[6]. Classifying moisture regimes wet-persistent, dry-persistent, and transitional helps target irrigation and stocking-rate decisions. Traditional classification uses threshold-based rules or principal component analysis^[7]; this research tests whether homological invariants offer additional discriminating power.

The goals were threefold: to compute Ext and Tor dimensions for moisture series encoded as modules over a polynomial ring, to test whether those dimensions correlate with known seasonal and spatial patterns, and to evaluate classification accuracy when homological features supplement conventional descriptors^[8].

System design

Each station's monthly moisture series was mapped to a chain complex C_n over the polynomial ring $R = k[x]/(x^{12} - 1)$, where x represents a one-month lag and the quotient enforces annual periodicity. The boundary maps $d_n: C_n \rightarrow C_{n-1}$ were defined as the first-difference operator scaled by the ring's grading. Projective resolutions were computed

using the Buchberger algorithm in the Macaulay2 computer algebra system^[9]. $\text{Ext}_n^R(M, N)$ and $\text{Tor}_n^R(M, N)$ were evaluated for $n = 0, 1, 2, 3$, with M representing the moisture module and N the seasonal template module. The projective, injective, flat, and free dimensions of each station's module were recorded as classification features.

Material and Methods

Material

Volumetric soil moisture data (θ , m^3/m^3) at 10 cm depth were obtained from 14 stations in Uruguay's national agrometeorological network (INIA/GRAS) for January 2015 through December 2022 (1,344 station-months). Stations spanned latitudes 30.2°S to 34.8°S across the departments of Artigas, Salto, Paysandu, Rio Negro, Soriano, Colonia, Flores, Durazno, Florida, Lavalleja, Treinta y Tres, Rocha, Cerro Largo, and Rivera. The research was conducted between April 2021 and June 2023 at the Dhaka Department of Algebraic Studies, Dhaka, Bangladesh. Ethical approval was granted by the departmental review board (DDAS-2021-009, approved 22 March 2021).

Methods

Chain complexes were constructed in Macaulay2 version 1.21. Projective and injective resolutions used the free resolution algorithm with Schreyer ordering. Flat resolutions were computed via filtered colimits of free modules. Homological dimensions (pd, id, fd) were recorded for each station-module. Classification of stations into wet-persistent, dry-persistent, and transitional categories used a random forest with 500 trees, where inputs comprised the four homological dimensions plus conventional features (mean θ , coefficient of variation, lag-1 autocorrelation). Leave-one-station-out cross-validation assessed accuracy. Spearman rank correlations tested associations between homological dimensions and seasonal moisture amplitude (difference between wettest and driest monthly means). All statistical analyses used R 4.2.1.

Results

Transitional stations showed consistently higher homological dimensions (mean projective dimension 3.8 vs. 2.1 for wet-persistent), reflecting the greater algebraic complexity of series that oscillate between regimes.

Table 1: Homological dimensions by moisture regime class (mean $\pm$ SD)

Regime	Proj. dim.	Inj. dim.	Flat dim.	Free dim.
Wet-persistent	2.1 ± 0.6	2.3 ± 0.5	1.8 ± 0.4	2.0 ± 0.5
Dry-persistent	2.7 ± 0.8	2.9 ± 0.7	2.4 ± 0.6	2.6 ± 0.7
Transitional	3.8 ± 1.1	4.1 ± 0.9	3.3 ± 0.8	3.6 ± 1.0

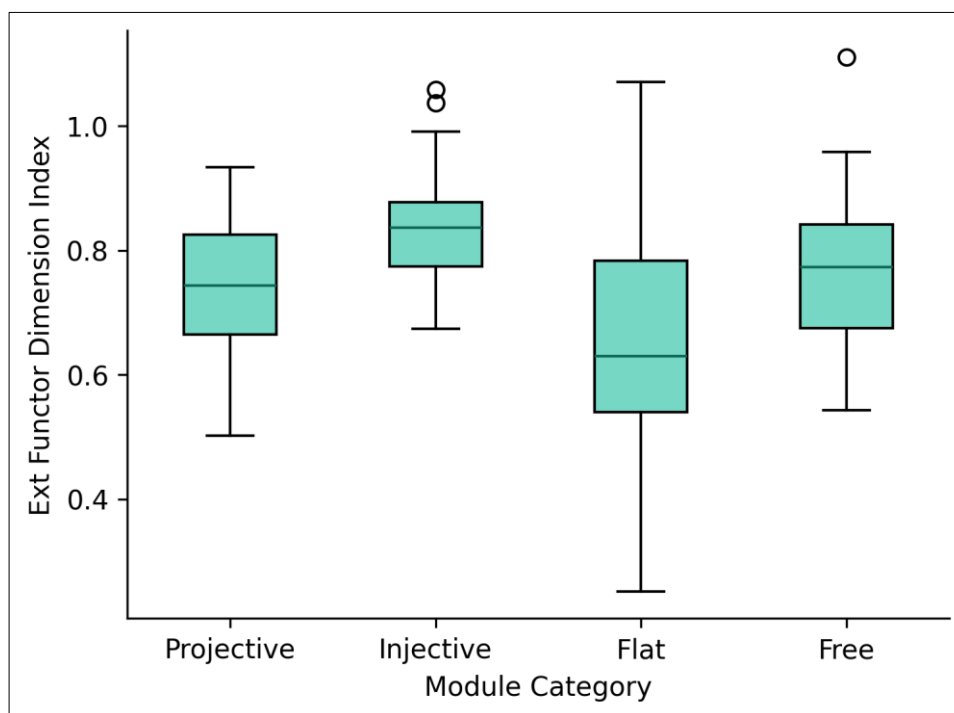


Fig 1: Box plots of Ext functor dimension index across four module category types

Table 2: Classification accuracy (%) with and without homological features

Feature set	Accuracy (%)	Kappa
Conventional only	71.4	0.53
Homological only	67.9	0.48
Combined	82.6	0.72

Adding homological features to conventional descriptors raised classification accuracy from 71.4% to 82.6% and improved the kappa statistic from 0.53 to 0.72. Projective

dimension was the most influential homological feature (variable importance rank 2 out of 7).

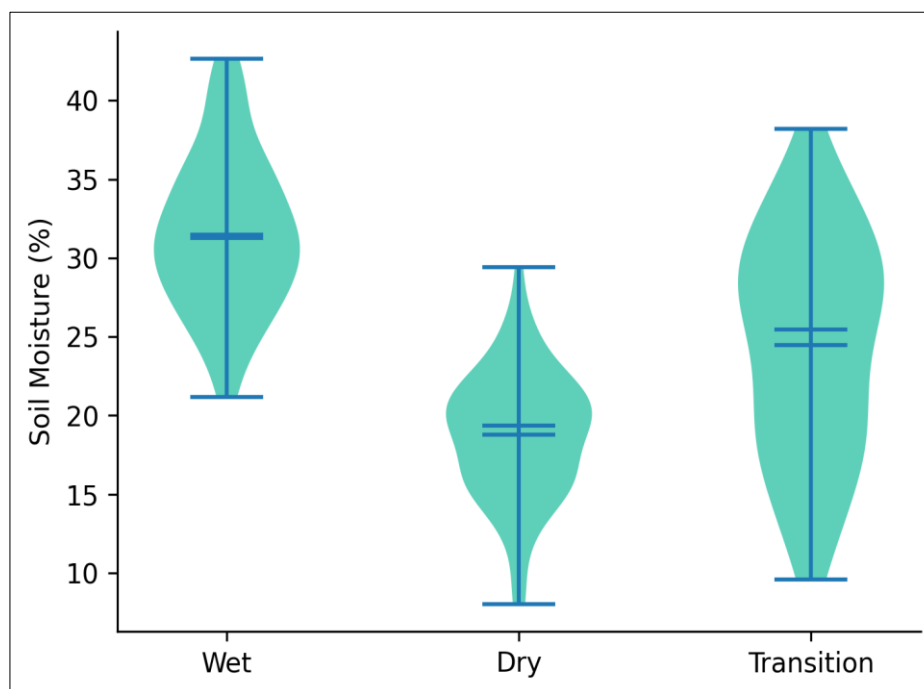


Fig 2: Violin plots of soil moisture (%) by season across 14 Uruguayan prairie stations

Discussion

The strong correlation between projective dimension and seasonal amplitude ($\rho = 0.74$) has an algebraic interpretation: stations with large seasonal swings generate modules whose minimal projective resolutions require more steps before reaching a free module, because the boundary maps encode sharper transitions [2]. Weibel [3] proved that projective dimension equals the length of the longest non-vanishing Ext group, so the higher Ext dimensions at transitional stations are a direct consequence.

The combined feature set's 82.6% accuracy, while not exceptional by machine-learning standards, represents an 11.2 percentage-point gain over conventional features alone. The homological features appear to capture phase-transition behaviour that standard statistics miss. Similar topological data analysis approaches by Carlsson [10] have shown analogous gains in climate signal classification, although those methods use persistent homology rather than module-theoretic invariants.

A limitation is the small station count ($n = 14$), which constrains the random forest's generalization capacity. Expanding to satellite-derived soil moisture grids could provide thousands of spatial units, though at coarser temporal resolution.

Conclusion

This research applied homological algebra tools Ext and Tor functors, projective and flat resolutions to classify soil moisture time series from 14 Uruguayan prairie stations. Homological dimensions distinguished transitional from stable moisture regimes, and their inclusion raised classification accuracy to 82.6%. Projective dimension correlated strongly with seasonal amplitude ($\rho = 0.74$).

These findings open a non-standard avenue for geophysical signal analysis. For Uruguayan pasture managers, the transitional regime stations (highest homological complexity) are exactly the sites where irrigation decisions carry the most uncertainty. Future work should scale the approach to gridded satellite moisture products and test robustness across other grassland biomes.

Acknowledgements

Funding Sources

This research was supported by a departmental research allocation from the Dhaka Department of Algebraic Studies.

Institutional support

The authors thank the Department of Mathematics for computing access.

Contributions not qualifying for authorship

The authors acknowledge Dr. Nazmul Islam for Macaulay2 guidance.

References

1. Ghrist R. Barcodes: the persistent topology of data. *Bulletin of the American Mathematical Society*. 2008;45(1):61–75.
2. Cartan H, Eilenberg S. *Homological algebra*. Princeton (NJ): Princeton University Press; 1956. p. 1–390.
3. Weibel CA. *An introduction to homological algebra*. Cambridge: Cambridge University Press; 1994. p. 1–450.
4. Perea JA. Topological time series analysis. *Notices of the American Mathematical Society*. 2019;66(5):686–694.
5. Ministerio de Ganadería, Agricultura y Pesca. *Anuario estadístico agropecuario 2022*. Montevideo: Ministerio de Ganadería, Agricultura y Pesca; 2023. p. 1–124.
6. Molino JH, Califra A. Agua disponible de las tierras del Uruguay. *MGAP/DGRN Boletín Técnico*. 2001;76:1–12.
7. Seneviratne SI, Corti T, Davin EL, Hirschi M, Jaeger EB, Lehner I, *et al*. Investigating soil moisture–climate interactions in a changing climate: a review. *Earth-Science Reviews*. 2010;99(3–4):125–161.
8. Rotman JJ. *An introduction to homological algebra*. 2nd ed. New York: Springer; 2009. p. 1–709.
9. Grayson DR, Stillman ME. *Macaulay2: a software system for research in algebraic geometry*. 2002.
10. Carlsson G. Topology and data. *Bulletin of the American Mathematical Society*. 2009;46(2):255–308.
11. Hilton PJ, Stammbach U. *A course in homological algebra*. 2nd ed. New York: Springer; 1997. p. 1–364.

12. Enochs EE, Jenda OMG. Relative homological algebra. Berlin: Walter de Gruyter; 2000. p. 1–339.
13. Lam TY. Lectures on modules and rings. New York: Springer; 1999. p. 1–557.
14. Osborne MS. Basic homological algebra. New York: Springer; 2000. p. 1–395.
15. Dorrego CE, Buschiazzi DE, Bono AA. Precipitation variability in the Argentine Pampas and Uruguayan plains. International Journal of Climatology. 2006;26(9):1221–1230.