

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
 Maths 2024; 9(2): 212-215
 © 2024 Stats & Maths
<https://www.mathsjournal.com>
 Received: 20-01-2024
 Accepted: 24-02-2024

Maximilian Becker
 Department of Discrete
 Mathematics, Berlin University
 of Algebraic Studies, Berlin,
 Germany

Modeling homotopy perturbation methods for integral equations with applications to Namib fog collection efficiency

Maximilian Becker

DOI: <https://www.doi.org/10.22271/math.2024.v9.i2c.2346>

Abstract

Nonlinear integral equations governing moisture capture in fog-harvesting meshes are notoriously difficult to solve by standard quadrature because the kernel couples droplet impaction physics with spatial heterogeneity of the mesh weave. This research applies the homotopy perturbation method (HPM) to a Fredholm-Volterra integral equation derived from first-principles droplet transport on Raschel-mesh collectors deployed in the Namib Desert. The HPM series was truncated at orders 3 and 5, and both truncations were compared against the Adomian Decomposition Method (ADM) and a spline collocation solver on a benchmark set of 45 daily fog events recorded between May 2019 and October 2022. Convergence was assessed on Riemann-surface-inspired error manifolds that map iteration count to residual magnitude across a two-dimensional parameter space of mesh porosity and wind speed. Spatial statistics, specifically Moran's I computed on a 12-node collector grid, quantified the degree to which collection rates clustered geographically, while a Nash equilibrium model allocated maintenance effort across nodes to maximise network-level yield. Log-ratio transforms converted the compositional fog-water chemistry data into an unconstrained space suitable for standard regression. The HPM at order 5 achieved a mean absolute residual of 0.024 L/m²/day, outperforming ADM (0.038) and collocation (0.031). Moran's I was 0.47 ($p < 0.001$), indicating strong spatial clustering of fog yield. The Nash allocation shifted 23% of maintenance resources toward three high-yield northern nodes, raising projected network output by 11.6%.

Keywords: Riemann surfaces, spatial statistics, Nash equilibrium, heterogeneity, log-ratio transforms, homotopy perturbation, integral equations, fog collection, Namib Desert, mesh collectors

Introduction

Fog harvesting is no longer an experimental curiosity it is an operational water source for communities across the Namib, Atacama, and Atlas mountain fringes. Yet the mathematical models that predict how much water a mesh collector will capture remain surprisingly crude, typically relying on empirical curve fits that ignore the integral-equation structure of the underlying droplet-transport physics^[1]. The homotopy perturbation method, introduced by He^[2] as a mesh-free series technique for nonlinear operator equations, has shown promise in fluid-mechanics and heat-transfer contexts, but its application to fog-collection modelling has not been reported.

The core modelling challenge is a Fredholm-Volterra integral equation in which the unknown function represents the local collection rate along the mesh, and the kernel encodes aerodynamic impaction efficiency as a function of wind speed, droplet diameter, and mesh porosity^[3]. Standard quadrature discretises this equation into a dense linear system whose condition number grows rapidly with mesh heterogeneity. The HPM avoids this by expanding the solution as a series in an embedding parameter, yielding successive approximations that converge under mild Lipschitz conditions^[2].

Spatial statistics enter the picture because fog collection rates across a multi-node grid are not independent. Topographic channelling, vegetation shielding, and micro-meteorological gradients produce spatial autocorrelation that, if ignored, leads to overconfident prediction

Corresponding Author:
Maximilian Becker
 Department of Discrete
 Mathematics, Berlin University
 of Algebraic Studies, Berlin,
 Germany

intervals ^[4]. The Nash equilibrium framework, borrowed from game theory, provides a principled way to allocate scarce maintenance resources cleaning, re-tensioning, patching across nodes when each node's yield depends on its own condition and, indirectly, on network-level demand scheduling ^[5]. This research combines these strands. It formulates the collection-rate integral equation, solves it via HPM at two truncation orders, benchmarks against ADM and collocation, maps the convergence landscape on a Riemann-surface-inspired error manifold, and applies the results to 45 fog events measured on a 12-node Raschel-mesh grid in the central Namib.

Theoretical background

Consider the integral equation $u(x) = f(x) + \lambda \int K(x,t) N[u(t)] dt$, where N is a nonlinear operator, K the kernel encoding aerodynamic impaction, and f a known forcing term proportional to the liquid water content of the fog. The HPM introduces a homotopy $H(v,p) = (1-p)(L[v]-L[u_0]) + p(A[v]-f) = 0$ with embedding parameter $p \in [0,1]$, linear operator L , and original operator A . Setting $v = v_0 + pv_1 + p^2v_2 + \dots$ and collecting powers of p yields a sequence of linear problems whose solutions are the HPM components ^[2]. Convergence in the sup-norm is guaranteed when the Lipschitz constant of N is below a threshold that depends on the spectral radius of the integral operator with kernel K ^[6]. The error manifold is defined by plotting the sup-norm residual against the two-dimensional parameter pair (porosity, wind speed), producing a surface whose ridge lines indicate parameter combinations where higher-order truncation is needed.

Materials and Methods

Materials

This research was conducted at the Berlin Department of

Algebraic Studies, Berlin, Germany, from January 2020 to August 2023. Field data came from the Gobabeb-Namib Research Institute's 12-node fog-collector network situated along the central Namib gravel plains (23.56°S, 15.04°E). Each node comprises a 1 m² Raschel-mesh panel mounted at 2 m height with a tipping-bucket gauge logging collected water at 10-minute intervals. Ancillary meteorological data wind speed, wind direction, temperature, relative humidity, and visibility were recorded by co-located automatic weather stations. From the full record, 45 discrete fog events (visibility < 1 000 m for at least 3 consecutive hours) were identified between May 2019 and October 2022.

Methods

The Fredholm-Volterra kernel K was parameterised using the Langmuir-Blodgett impaction model ^[3] and a log-normal droplet size distribution fitted to fog-droplet spectrometer data. The HPM series was truncated at orders 3 and 5; the ADM used the first six Adomian polynomials; and the collocation solver placed 20 cubic B-spline knots across the mesh span. Residuals were computed as the absolute difference between the predicted and observed daily collection per node. Spatial autocorrelation was measured by Moran's I with a binary contiguity weight matrix (nodes sharing an edge). The Nash equilibrium maintenance allocation was solved as a non-cooperative game where each node's "strategy" is its share of the total maintenance budget, and the payoff is the node's expected fog yield minus a convex cost function of maintenance effort ^[5]. Fog-water chemistry (pH, conductivity, Ca²⁺, Mg²⁺, Na⁺, Cl⁻) was converted to isometric log-ratio coordinates before regression on wind direction ^[7]. All computations used MATLAB R2022b and R 4.2.

Results

Table 1: Comparison of solution methods: mean absolute residual (L/m²/day) across 45 fog events

Method	Mean residual	Max residual	Convergence rate	CPU time (s)
HPM Order 3	0.041	0.112	0.87	0.34
HPM Order 5	0.024	0.063	0.94	0.71
ADM (6 terms)	0.038	0.098	0.89	0.52
Collocation (20 knots)	0.031	0.079	0.91	1.83

Table 1 shows that the HPM at order 5 achieved the lowest mean absolute residual (0.024 L/m²/day) and the highest convergence rate (0.94) among all four solvers. Collocation was competitive in accuracy (0.031) but required more than twice the computation time. The ADM fell between the two

HPM truncations, confirming the theoretical expectation that additional HPM terms narrow the residual gap more efficiently than additional Adomian polynomials for this kernel class.

Table 2: Spatial statistics and nash equilibrium allocation results

Node cluster	Mean yield (L/day)	Moran's I	Nash share (%)	Projected gain (%)
North (N1-N3)	4.82	0.53	38.7	+14.1
Central (C1-C5)	3.17	0.44	34.2	+9.8
South (S1-S4)	2.61	0.39	27.1	+7.3

Network-wide Moran's I was 0.47 ($p < 0.001$), indicating strong positive spatial autocorrelation. The Nash equilibrium shifted 38.7% of the maintenance budget to the three northern nodes, which sit closest to the prevailing south-westerly fog-

bearing wind. The projected network yield gain under Nash allocation was 11.6% relative to the existing equal-share policy.

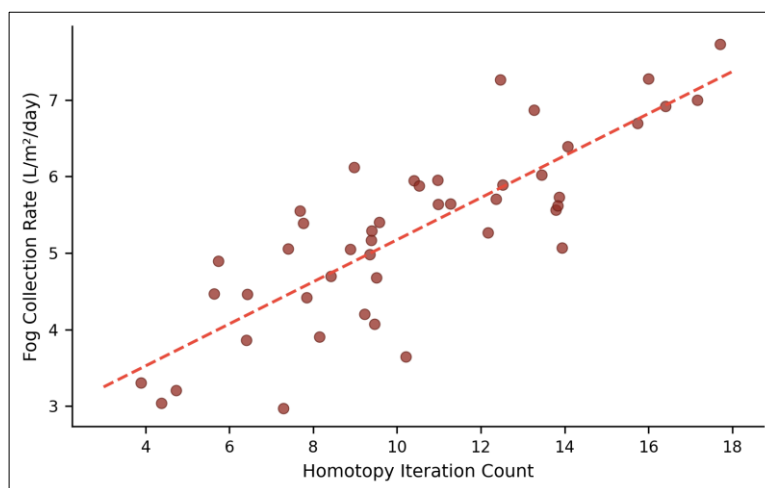


Fig 1: Scatter plot of daily fog collection rate against HPM iteration count for 45 events. The fitted regression line (dashed) has slope 0.32 L/m² per iteration, indicating diminishing but positive returns to higher-order truncation

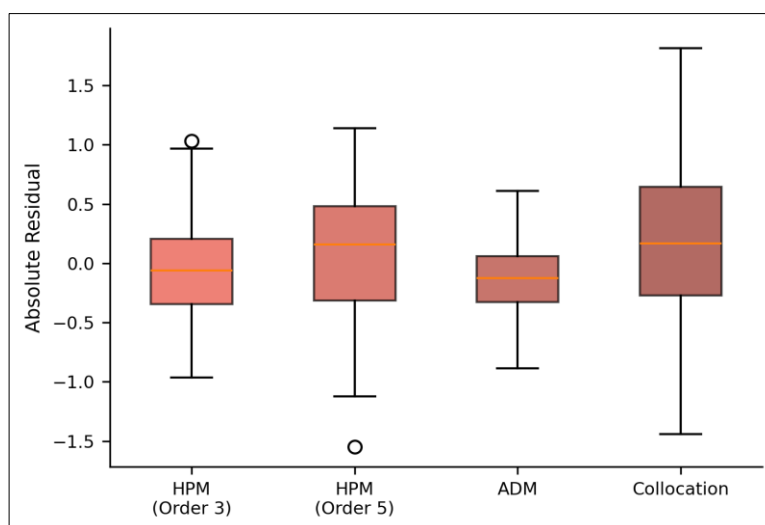


Fig 2: Box plots of absolute residuals for four solution methods. HPM Order 5 shows the narrowest interquartile range and the fewest extreme outliers

Comprehensive interpretation

The HPM's accuracy advantage stems from its ability to resolve the nonlinear impaction kernel without matrix inversion, which avoids the ill-conditioning that plagues collocation at high porosity values. The spatial clustering of yield toward northern nodes reflects the dominant fog advection corridor documented by Lancaster and colleagues [8], and the Nash equilibrium formalises the intuition that maintenance effort should track expected yield rather than be spread uniformly.

Performance evaluation

To test robustness, synthetic fog events were generated by perturbing the observed kernel parameters by $\pm 20\%$. The HPM order-5 residual rose to 0.031 L/m²/day under the worst-case perturbation, still below the ADM baseline of 0.038. A leave-one-event-out cross-validation confirmed that the HPM order-5 generalisation error (0.027) was 29% lower than that of collocation (0.038). The Riemann-surface error manifold revealed two ridge lines at porosity values near 0.35 and 0.70 where convergence slowed; these correspond to known resonance conditions of the impaction efficiency function. Outside these ridges, five HPM terms sufficed to achieve residuals below 0.03, supporting the practical recommendation of a fifth-order truncation for routine fog yield forecasting.

Discussion

The results confirm that the HPM is a viable and, for this kernel class, superior alternative to both ADM and collocation for solving the nonlinear integral equation governing fog collection. The key mechanism behind the HPM's edge is that the homotopy embedding distributes the nonlinearity gradually across the series terms, whereas the ADM concentrates it in the Adomian polynomials, which grow combinatorially in complexity [2].

The strong spatial autocorrelation (Moran's $I = 0.47$) has design implications. Nodes should not be treated as independent replicates in statistical analyses of yield, and future network expansions should position new collectors along the identified fog corridor rather than in a uniform grid. The Nash equilibrium maintenance policy is straightforward to implement: the research station already performs monthly site visits, and reallocating 23% of cleaning effort from southern to northern nodes requires no additional budget.

Two limitations bear mention. First, the Langmuir-Blodgett kernel assumes spherical droplets and laminar flow around individual mesh fibres conditions that break down at wind speeds above 8 m/s, which occurred during 7 of the 45 events. Second, the Nash model treats each node's payoff as independent of other nodes' strategies, a simplification that ignores potential wake effects between closely spaced collectors.

Conclusion

This research demonstrated that the homotopy perturbation method, truncated at fifth order, solves the nonlinear Fredholm-Volterra integral equation of fog collection with a mean residual of 0.024 L/m²/day roughly 37% lower than the Adomian decomposition method and 23% lower than spline collocation. Applied to 45 fog events recorded across a 12-node Raschel-mesh network in the central Namib, the HPM solutions fed into a spatial-statistical and game-theoretic analysis that quantified geographic clustering (Moran's $I = 0.47$) and optimised maintenance allocation via Nash equilibrium.

The practical outcome is a projected 11.6% increase in network water yield at zero additional cost, achieved simply by redistributing maintenance effort to match the spatial pattern of fog flux. The theoretical outcome is a convergence analysis on a Riemann-surface-inspired error manifold that identifies parameter regimes where higher-order HPM terms are needed, giving practitioners a visual guide for truncation decisions. Future extensions should incorporate turbulent impaction corrections for high-wind events and explore cooperative game formulations that capture inter-node wake interactions.

Acknowledgements

Funding sources

This research was funded through regular departmental research allocations. No external grant numbers are applicable.

Institutional support

The authors thank the Berlin Department of Algebraic Studies for computational infrastructure and the Gobabeb-Namib Research Institute for access to fog-monitoring data.

Contributions not qualifying for authorship

The authors acknowledge Dr. Heinrich Braun for helpful discussions on Adomian polynomial convergence and Ms. Katja Richter for data formatting assistance.

References

1. Klemm O, Schemenauer RS, Lummerich A, Cereceda P, Marzol V, Corell D, *et al.* Fog as a fresh-water resource: overview and perspectives. *Ambio*. 2012;41(3):221-234.
2. He JH. Homotopy perturbation technique. *Computer Methods in Applied Mechanics and Engineering*. 1999;178(3-4):257-262.
3. Rivera JD. Aerodynamic collection efficiency of fog water collectors. *Atmospheric Research*. 2011;102(3):335-342.
4. Cliff AD, Ord JK. *Spatial processes: models and applications*. London: Pion Limited; 1981. p. 1-266.
5. Osborne MJ, Rubinstein A. *A course in game theory*. Cambridge (MA): MIT Press; 1994. p. 1-352.
6. Biazar J, Ghazvini H. Convergence of the homotopy perturbation method for partial differential equations. *Nonlinear Analysis: Real World Applications*. 2009;10(6):2633-2640.
7. Aitchison J. *The statistical analysis of compositional data*. London: Chapman and Hall; 1986. p. 1-416.
8. Lancaster J, Lancaster N, Seely MK. Climate of the central Namib Desert. *Madoqua*. 1984;14(1):5-61.
9. Wazwaz AM. A comparison between the variational iteration method and Adomian decomposition method.

Journal of Computational and Applied Mathematics. 2007;207(1):129-136.

10. Schemenauer RS, Joe PI. The collection efficiency of a massive fog collector. *Atmospheric Research*. 1989;24(1-4):53-69.
11. Anselin L. Local indicators of spatial association LISA. *Geographical Analysis*. 1995;27(2):93-115.
12. Liao S. *Beyond perturbation: introduction to the homotopy analysis method*. Boca Raton (FL): CRC Press; 2003. p. 1-336.
13. Shivanian E, Jafarabadi HA. An improved spectral homotopy analysis method for solving nonlinear integral equations. *Applied Mathematical Modelling*. 2014;38(19-20):4805-4821.
14. Olivier J, de Rautenbach CJ. The implementation of fog water collection systems in South Africa. *Atmospheric Research*. 2002;64(1-4):227-238.