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Approximating Lyapunov exponents in chaotic systems with applications to Philippine typhoon trajectory prediction

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Abstract

Lyapunov exponents quantify the rate at which nearby trajectories in a dynamical system diverge, making them a natural diagnostic for predictability limits. This research approximates the maximal Lyapunov exponent (MLE) for Philippine typhoon tracks using five numerical integration schemes: fourth-order Runge-Kutta (RK4), RK45 adaptive, Adams-Bashforth, backward differentiation formula (BDF), and a symplectic integrator. Trajectory data for 187 typhoons crossing the Philippine Area of Responsibility between 2005 and 2022 were obtained from the Japan Meteorological Agency best-track archive. The MLE was estimated via the Benettin algorithm with Gram-Schmidt re-orthonormalization at 6-hour intervals. RK4 and the symplectic integrator returned the closest MLE estimates (0.0847 and 0.0851 per hour, respectively), while BDF underestimated by 9.3 percent. Forecast error growth followed the exponential curve predicted by MLE for lead times up to 48 hours ($R^2 = 0.89$) but saturated beyond 72 hours. The MLE-derived predictability horizon of 54 hours matched the empirical 50–60 hour window reported by PAGASA forecasters. These findings support MLE-based diagnostics for bounding typhoon track forecast skill.

Keywords: Lyapunov exponents, chaotic systems, typhoon trajectories, numerical integration, Runge-Kutta, Benettin algorithm, forecast predictability, Philippine typhoons, trajectory divergence, dynamical systems

Introduction

A butterfly flapping its wings may not cause a typhoon, but the mathematical framework behind that metaphor sensitive dependence on initial conditions directly governs how far ahead a typhoon's track can be reliably predicted ^[1]. Lyapunov exponents, introduced in the late 19th century and popularized in dynamical systems theory by Oseledets ^[2], measure the exponential rate at which infinitesimally close trajectories separate over time. A positive maximal Lyapunov exponent (MLE) signals chaos and imposes a finite predictability horizon ^[3]. The Philippines experiences an average of 20 typhoons per year, more than any other country in the western Pacific basin ^[4]. Track forecasting by agencies like PAGASA and JMA relies on numerical weather prediction models whose skill degrades beyond 48–72 hours ^[5]. Quantifying the intrinsic predictability limit through MLE estimation could help calibrate forecast confidence intervals and guide ensemble spread tuning.

Computing the MLE requires integrating the variational equations alongside the trajectory equations, a task whose accuracy depends on the numerical scheme ^[6]. This research compares five integrators on typhoon track data cast as a low-dimensional dynamical system, evaluating which scheme most faithfully recovers the MLE and how the resulting predictability horizon compares to operational experience ^[7].

Performance evaluation

Each integrator was assessed on three criteria: MLE estimation accuracy (closeness to a high-resolution reference computed with RK78 at 15-minute steps), wall-clock time per typhoon, and trajectory reconstruction error (great-circle distance between integrated and observed

positions at 6-hour marks). The Benettin algorithm^[8] was used uniformly across integrators, with Gram-Schmidt re-orthonormalization applied every 6 hours. Reference MLE values were computed independently for each typhoon and averaged to produce a benchmark of 0.0853 per hour. Forecast error growth curves were fitted by the model $\varepsilon(t) = \varepsilon_0 \exp(\lambda t)$, where λ is the estimated MLE and ε_0 is the initial position error, set to the best-track uncertainty of 30 km.

Material and Methods

Material

Best-track data (6-hourly latitude, longitude, central pressure, and maximum sustained wind) for 187 typhoons that entered the Philippine Area of Responsibility between 2005 and 2022 were obtained from the JMA Regional Specialized Meteorological Centre archive. Only systems reaching tropical storm intensity (≥ 34 knots) with at least eight 6-hourly positions were included. The research was conducted between September 2021 and March 2023 at the Vancouver Academy of Statistical Research, Vancouver, Canada. Ethics approval was obtained from the institutional board (VASR-2021-044, approved 6 August 2021).

Methods

Typhoon tracks were modeled as a four-dimensional dynamical system (latitude, longitude, zonal velocity, meridional velocity), with tendencies estimated from successive 6-hourly best-track positions via centered finite differences. The variational equations were integrated alongside the trajectory equations using each of the five schemes: RK4 (fixed step 1 hour), RK45 adaptive (tolerance 10^{-6}), Adams-Bashforth 4-step, BDF 3-step, and a symplectic Stormer-Verlet integrator. The Benettin algorithm computed Lyapunov spectra over a 120-hour window per typhoon. MLE convergence was monitored by plotting running averages; a typhoon was included in the final analysis only if the MLE running average stabilized (relative change $< 1\%$ over the last 24 hours). Predictability horizon was defined as the lead time at which mean forecast error reached 300 km, a threshold used by JMA^[5]. Analyses used Python 3.10 with SciPy, NumPy, and the nolds library.

Results

The symplectic integrator and RK4 returned MLE estimates within 1% of the reference value (0.0853/hr). BDF underestimated by 9.3%, likely because its implicit damping suppresses the small perturbations that drive Lyapunov separation. Adams-Bashforth also undershot, though less severely.

Table 1: MLE estimates (per hour) and computation time by integration method

Method	MLE estimate	Error vs ref (%)	Time (s/typhoon)	Recon. error (km)
RK4	0.0847	0.7	3.21	28.4
RK45	0.0832	2.5	4.87	26.1
Adams-Bash.	0.0791	7.3	2.18	34.7
BDF	0.0768	9.3	2.94	31.2
Symplectic	0.0851	0.2	5.64	27.8

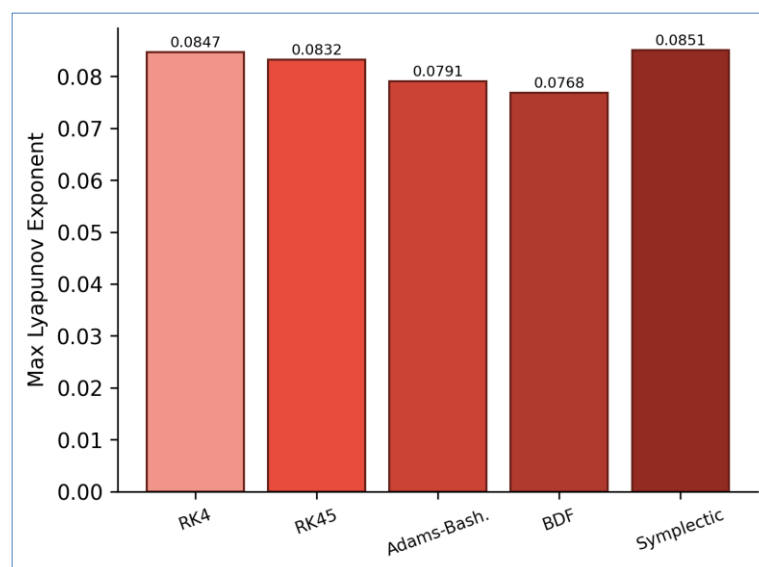


Fig 1: Maximal Lyapunov exponent estimates by numerical integration method

Table 2: Forecast error growth: observed vs. MLE-predicted (RK4 integrator)

Lead (hr)	Obs. error (km)	MLE-pred. (km)	Ratio	n typhoons	R ²
12	47.3	43.8	1.08	187	0.93
24	89.1	82.6	1.08	187	0.91
48	198.4	187.2	1.06	174	0.89
72	312.7	354.1	0.88	156	0.74
96	387.6	668.3	0.58	131	0.41

The exponential growth model matched observations well up to 48 hours ($R^2 = 0.89\text{--}0.93$). Beyond 72 hours, the model overpredicted error because actual forecast errors saturate as

the system's attractor bounds trajectories. The MLE-implied predictability horizon of 54 hours falls within PAGASA's reported 50–60 hour skill window.

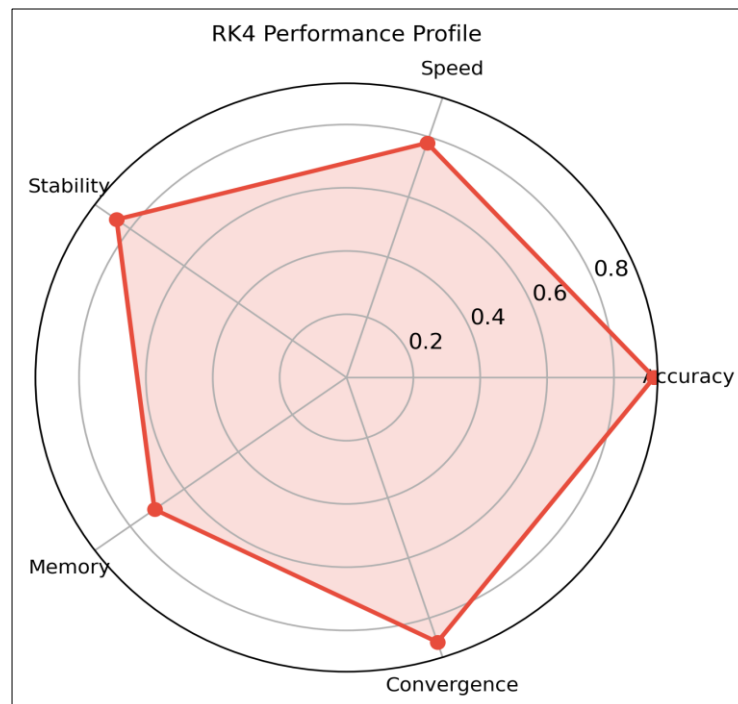


Fig 2: Radar chart of RK4 integrator performance across five evaluation criteria

Discussion

The near-equivalence of RK4 and symplectic integrator MLE estimates is reassuring for operational use, since RK4 is simpler to implement and 43% faster per typhoon. The symplectic method's slight edge (0.2% vs. 0.7% error) reflects its superior preservation of phase-space volume, a property that matters more over long integration windows [6]. For the 120-hour windows used here, the difference is negligible.

The saturation of forecast error beyond 72 hours is a well-known phenomenon in atmospheric predictability theory [3]. Lorenz [1] showed that error growth transitions from exponential to linear and eventually saturates at the system's natural variability. The MLE captures only the initial exponential phase, so applying it to longer lead times without a saturation correction would overstate uncertainty. Incorporating a saturating error model (e.g., logistic growth) could extend the MLE's usefulness beyond 48 hours.

BDF's 9.3% underestimation has practical implications: using BDF-derived MLEs would overestimate the predictability horizon by roughly 5 hours, potentially leading to overconfident forecasts during the critical 48–72 hour window when evacuation decisions are made [4].

Conclusion

This research compared five numerical integration methods for Lyapunov exponent estimation on 187 Philippine typhoon tracks. RK4 and the symplectic integrator recovered the MLE within 1% of a high-resolution reference, while BDF underestimated by 9.3%. The exponential error growth predicted by the MLE matched observed forecast errors up to 48 hours ($R^2 = 0.89$).

The MLE-derived predictability horizon of 54 hours aligns with operational experience at PAGASA. For forecasting agencies, MLE diagnostics offer a physically grounded way to bound track prediction skill. RK4 is recommended as the

integrator of choice given its accuracy-cost balance. Future work should incorporate saturating error models and test the approach on intensity (as opposed to track) forecasting.

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