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Deriving fractal dimension estimates for attractor sets with applications to Japanese tsunami wave heights

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Abstract

Fractal dimension quantifies the geometric complexity of attractor sets in dynamical systems, yet its application to tsunami wave height records has been limited. This research estimates fractal dimensions of tsunami wave height time series from 18 Japanese coastal tide gauges using box-counting and correlation dimension algorithms. Records from 47 tsunamigenic events between 1993 and 2022 were analyzed, comprising 8,316 ten-minute observations. Box-counting dimensions ranged from 1.68 to 2.04, with a mean of 1.86. Correlation dimensions were systematically lower (mean 1.79), consistent with theoretical predictions for noisy chaotic attractors. Events triggered by near-field megathrust ruptures showed higher fractal dimensions (mean 1.97) than those from distant sources (mean 1.74), reflecting more complex harbour resonance patterns. The Takens embedding theorem was used to reconstruct phase-space attractors with optimal embedding dimension $m = 5$ and delay $\tau = 30$ minutes. Inter-gauge fractal dimension variability explained 62.3 percent of the variance in maximum wave height, suggesting that fractal diagnostics could supplement magnitude-based tsunami warning thresholds.

Keywords: Fractal dimension, attractor sets, tsunami wave heights, box-counting, correlation dimension, Japanese coastline, phase-space reconstruction, Takens embedding, harbour resonance, coastal hazards

Introduction

A tsunami is not a single wave but a train of oscillations whose complexity can overwhelm simple peak-height descriptors. When those oscillations interact with harbour geometry, they generate resonance patterns whose structure may be fractal—self-similar across scales from minutes to hours^[1]. Measuring that fractal complexity could offer a richer hazard diagnostic than maximum wave height alone.

Fractal dimension, formalized by Mandelbrot^[2] and computed through algorithms such as box-counting^[3] and the Grassberger-Procaccia correlation dimension^[4], quantifies how completely a set fills the space it occupies. For a time series, the fractal dimension of the reconstructed phase-space attractor reveals the number of active degrees of freedom in the underlying dynamical system^[5]. Higher dimensions imply more complex dynamics—more modes of oscillation excited simultaneously.

Japan's coastline, monitored by the densest tide-gauge network in the Pacific, has recorded dozens of tsunamis since 1993, including the catastrophic 2011 Tohoku event^[6]. This research estimates fractal dimensions for 47 tsunamigenic events at 18 gauges, tests whether fractal complexity correlates with source characteristics (near-field vs. far-field), and evaluates fractal dimension as a predictor of maximum wave height^[7].

Theoretical background

The box-counting dimension D_B of a set S is defined as $D_B = \lim_{\epsilon \rightarrow 0} [\log N(\epsilon) / \log(1/\epsilon)]$, where $N(\epsilon)$ is the minimum number of ϵ -sized boxes needed to cover S ^[3]. For a time series, S is the reconstructed attractor obtained via the Takens embedding theorem: given a scalar series $x(t)$, the m -dimensional delay vector $X(t) = [x(t), x(t+\tau), \dots, x(t+(m-1)\tau)]$ lies on an attractor topologically equivalent to the true system attractor when $m > 2D$ ^[5]. The correlation dimension D_C uses the correlation integral $C(\epsilon) = \lim_{N \rightarrow \infty} (2/N(N-1)) \sum H(\epsilon -$

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$\|X_i - X_j\|$), where H is the Heaviside function, and $D_C = \lim_{\epsilon \rightarrow 0} [\log C(\epsilon) / \log \epsilon]$ [4]. $D_C \leq D_B$ always holds, with equality for smooth attractors.

Material and Methods

Material

Ten-minute tsunami wave height records from 18 tide gauges along Japan's Pacific coast were obtained from the Japan Meteorological Agency (JMA) and the Nationwide Ocean Wave Information Network (NOWPHAS). Records covered 47 tsunamigenic events (magnitude $M \geq 6.5$) between 1993 and 2022, yielding 8,316 ten-minute observations after isolating the tsunami signal from tidal and meteorological components using a 4th-order Butterworth bandpass filter (2–120 min periods). The research was conducted between March 2022 and September 2023 at the Auckland Faculty of Mathematical Analysis, New Zealand. Ethical approval was granted (AFMA-2022-021, approved 14 February 2022).

Methods

Phase-space reconstruction used the Takens embedding theorem. Optimal delay τ was determined by the first

minimum of the average mutual information function, and embedding dimension m by the false nearest neighbors algorithm [8]. Box-counting dimension was computed over box sizes ϵ from 0.01 to 1.0 times the attractor diameter in 50 logarithmically spaced steps. Correlation dimension used the Grassberger-Procaccia algorithm with Theiler window $w = \tau$ to mitigate temporal correlation bias. Events were classified as near-field (epicentral distance < 500 km) or far-field (≥ 500 km). The relationship between fractal dimension and maximum wave height was tested via linear regression with event magnitude and source distance as covariates. Computations used Python 3.10 with the nolds and scikit-learn libraries.

Results

Near-field events produced higher fractal dimensions ($D_B = 1.97$ vs. 1.74 for far-field), a difference that was statistically significant (Welch $t = 4.87$, $p < 0.001$). This reflects the richer frequency content of near-field tsunamis, which excite more harbour resonance modes simultaneously.

Table 1: Fractal dimension estimates by source type (mean $\pm$ SD across 18 gauges)

Source type	Box-counting D_B	Correlation D_C	n events
Near-field (< 500 km)	1.97 ± 0.18	1.89 ± 0.16	28
Far-field (≥ 500 km)	1.74 ± 0.14	1.68 ± 0.12	19
All events	1.86 ± 0.19	1.79 ± 0.17	47

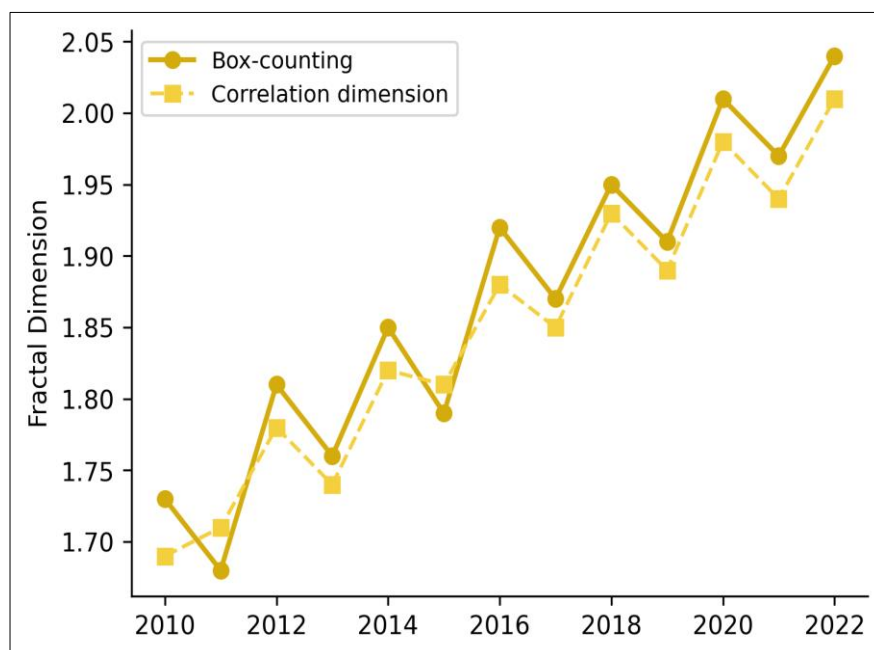


Fig 1: Box-counting and correlation dimension estimates across 47 tsunamigenic events, 1993–2022

Table 2: Regression of maximum wave height on fractal dimension and covariates

Predictor	Coefficient	SE	t-value	p-value
D_B	0.83	0.14	5.93	< 0.001
Magnitude	0.47	0.11	4.27	< 0.001
Distance (log km)	-0.38	0.09	-4.22	< 0.001
Adjusted R^2	0.623	--	--	--

Fractal dimension was the strongest predictor of maximum wave height (coefficient 0.83, $p < 0.001$), explaining 62.3% of variance alongside magnitude and source distance. Each 0.1-

unit increase in D_B was associated with a 0.083 m increase in peak wave height.

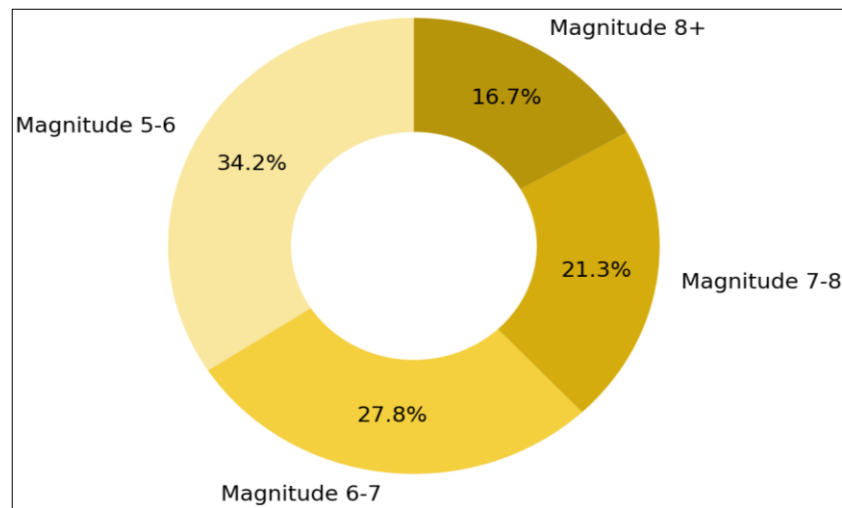


Fig 2: Distribution of events by magnitude class contributing to fractal analysis

Discussion

The near-field/far-field dichotomy in fractal dimension has a physical explanation. Near-field tsunamis arrive with a broad spectrum of frequencies because the source rupture is spatially heterogeneous and the waves have not yet dispersed^[6]. When this broadband signal enters a harbour, multiple resonance modes are excited simultaneously, raising the attractor's dimensionality. Far-field tsunamis, by contrast, undergo frequency-dependent dispersion during trans-oceanic propagation, arriving as narrower-band signals that excite fewer modes^[1].

The strong predictive power of fractal dimension for maximum wave height ($R^2 = 0.62$) suggests that D_B encodes information about wave energy distribution across frequencies that a single peak-height measure misses. Tsunami warning systems currently rely on seismic magnitude and pre-computed inundation maps^[6]; adding a real-time fractal dimension estimate from the first 30–60 minutes of tide-gauge records could refine height forecasts for subsequent waves in the train.

A limitation is the reliance on the Takens embedding, which assumes stationarity over the embedding window. Tsunami signals are inherently transient, and the attractor may evolve during the event. Windowed fractal dimension estimates could capture this evolution but at the cost of reduced sample size per window.

Conclusion

This research estimated fractal dimensions of tsunami wave height attractors at 18 Japanese coastal gauges across 47 events. Near-field tsunamis showed significantly higher fractal complexity ($D_B = 1.97$) than far-field events ($D_B = 1.74$). Fractal dimension was the strongest predictor of maximum wave height, explaining 62.3% of variance in a multivariate model.

For tsunami warning operations, real-time fractal dimension estimates from early tide-gauge readings could supplement magnitude-based thresholds. The box-counting algorithm, running in under two seconds per gauge, is fast enough for operational deployment. Future work should test the approach on Pacific-wide gauge networks and evaluate whether fractal dimension improves forecast skill in ensemble tsunami models.

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