

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2024; 9(2): 204-207
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<https://www.mathsjournal.com>
Received: 19-01-2024
Accepted: 23-02-2024

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Establishing asymptotic behavior of solutions to parabolic equations with applications to Gobi oasis water table data

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DOI: <https://www.doi.org/10.22271/math.2024.v9.i2c.2344>

Abstract

What mechanisms govern the long-term behavior of temperature-driven diffusion in arid groundwater systems? This research addresses that question by establishing asymptotic bounds for solutions to a class of semilinear parabolic Partial Differential Equations (PDEs), then applying those bounds to water table measurements from five oasis sites across the Gobi Desert. The mathematical framework combines Fourier-mode decomposition with energy-functional techniques to derive explicit decay rates for the solution operator in weighted Sobolev spaces. A cluster sampling protocol was adopted to collect monthly water table readings from January 2019 through August 2023 at depths ranging from 3.87 m to 8.14 m below surface level. The sampled data were fitted to a Generalized Extreme Value (GEV) distribution, and the resulting shape parameters were compared against the theoretical decay profile predicted by the PDE model. Neural network regression supplemented the classical analysis by capturing nonlinear deviations between observed drawdown curves and analytical predictions. Results indicate that the analytical decay envelope bounded 91.3% of observed trajectories within a 95% confidence band, while the neural network corrector reduced the root-mean-square residual from 0.37 m to 0.11 m. The GEV tail index aligned with the algebraic decay exponent ($p = 0.024$), supporting the hypothesis that extreme water table fluctuations follow the same power-law relaxation as the PDE solution. These findings bridge abstract regularity theory and field-level hydrology, offering a template for applying functional-analytic tools to environmental monitoring datasets.

Keywords: Partial differential equations, Fourier analysis, cluster sampling, neural networks, gev distribution, asymptotic decay, sobolev spaces, groundwater modeling, parabolic operators, arid hydrology

Introduction

How reliably can abstract convergence theorems predict the fate of groundwater beneath a shrinking desert oasis? The question sits at the intersection of pure analysis and applied earth science, yet it has received surprisingly little direct attention. Parabolic PDEs have long served as the workhorse model for heat and mass transfer processes^[1], and their asymptotic properties particularly exponential and algebraic decay of solutions are well mapped in the theoretical literature^[2]. At the same time, hydrologists tracking the Gobi Desert's retreating water tables rely on empirical time-series tools that seldom invoke the full power of PDE regularity results^[3].

Several authors have examined decay estimates for linear parabolic equations in bounded domains using semigroup methods^[4] or spectral decomposition^[5]. Extensions to semilinear problems appeared in the early 2000s, driven by reaction-diffusion applications^[6]. But translating those estimates into something a field hydrologist can test against borehole data isn't straightforward, because real aquifers introduce spatial heterogeneity, seasonal forcing, and measurement noise that the classical theory ignores.

On the data side, cluster sampling has become the preferred design for large-area groundwater surveys because it reduces travel cost while preserving spatial coverage^[7]. Fitting extreme-value distributions to water table minima is standard practice in drought-risk assessment^[8].

and the GEV family Gumbel, Fréchet, Weibull offers flexible tail behavior indexed by a single shape parameter^[9]. What has been missing is a clear mathematical link between the PDE decay exponent and the GEV shape parameter, a link this paper aims to establish.

Neural networks enter the picture as a practical correction layer. Universal approximation results guarantee that a sufficiently wide feed-forward network can capture residual nonlinearity left after the analytical model has been subtracted^[10]. Previous work applied neural surrogates to groundwater simulation in alluvial plains^[11], but not specifically to test asymptotic decay hypotheses in arid basins.

This research pursues three objectives. First, it derives sharp asymptotic upper bounds for a semilinear parabolic equation whose diffusion coefficient reflects depth-dependent permeability. Second, it tests those bounds against a 56-month record of water table depths at five Gobi oasis sites collected through a two-stage cluster design. Third, it trains a shallow neural network on the residual field to quantify how much predictive accuracy the analytical bound alone leaves on the table. The combined analytical-statistical-machine-learning pipeline offers a reusable template for bridging PDE theory and environmental field data.

Theoretical background

The governing equation takes the form $\partial u/\partial t = \nabla \cdot (a(x)\nabla u) + f(u)$, posed on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ with homogeneous Dirichlet conditions. Here $a(x)$ encodes the hydraulic conductivity field and f is a Lipschitz nonlinearity satisfying $f(0) = 0$. Under these assumptions the operator generates an analytic semigroup on $L^2(\Omega)$, and classical results^[4] yield the exponential bound $\|u(t)\| \leq C e^{-\lambda_1 t} \|u_0\|$ for the linear case, where λ_1 is the principal eigenvalue of $-\nabla \cdot (a\nabla \cdot)$.

When f introduces sublinear growth, the decay rate slows to algebraic: $\|u(t)\| \leq C(1+t)^{-\alpha}$ with α depending on the growth exponent of f . The energy-functional method of Haraux and Jendoubi^[12] provides the sharpest known bounds for such problems. This research extends their framework by incorporating a spatially variable diffusion coefficient calibrated from field permeability tests, which modifies λ_1 and consequently shifts the decay envelope. The modified eigenvalue was computed numerically on a triangulated mesh of each oasis sub-basin, giving site-specific decay parameters that feed directly into the statistical

comparison described in the Methods section.

Materials and Methods

Materials

This research was conducted at the Chiang Mai Foundation for Mathematical Modeling, Chiang Mai, Thailand, between March 2019 and September 2023. Water table depth records were obtained from five monitoring stations operated by the China-Mongolia Arid Zone Groundwater Network across the Gobi Desert: Ejina Oasis (41.95°N, 101.07°E), Dunhuang Basin (40.14°N, 94.66°E), Heihe Corridor (38.93°N, 100.43°E), Badain Jaran (39.77°N, 102.42°E), and Minqin Oasis (38.63°N, 103.08°E). Each station logs depth to water at hourly intervals using vented pressure transducers (In-Situ Rugged TROLL 100). Monthly medians were computed, giving 56 observations per site and 280 records in total.

Hydraulic conductivity profiles were drawn from published borehole slug-test surveys^[13] covering the same aquifer units. Topographic and soil texture data came from the Harmonized World Soil Database v1.2, downscaled to 250 m resolution.

Methods

A two-stage cluster sampling design was adopted following the guidelines of Cochran^[7]. At the first stage, five oasis clusters were selected with probability proportional to estimated aquifer area. At the second stage, three boreholes per cluster were randomly chosen, and their monthly medians were averaged to produce a single site-level time series. The GEV distribution was fitted to annual minima extracted from each series using L-moment estimation^[14], yielding location (μ), scale (σ), and shape (ξ) parameters.

The analytical decay bound was evaluated by solving the eigenvalue problem on each site's conductivity field via a P1 finite-element discretization (FreeFEM++ v4.12). The predicted decay profile $\hat{u}(t)$ was then compared to the observed drawdown curve by computing the fraction of monthly readings falling within the analytical envelope, defined as the region between $\hat{u}(t)$ and $\hat{u}(t) + 2\sigma_{\text{GEV}}$.

A single-hidden-layer neural network with 20 sigmoidal units was trained on the residual $r(t) = u_{\text{obs}}(t) - \hat{u}(t)$ using 80% of the data for training and 20% for validation. Performance was measured by root-mean-square error (RMSE) and the coefficient of determination R^2 . All computations used Python 3.10 with NumPy, SciPy, and PyTorch 1.13.

Results

Table 1: GEV parameter estimates and analytical decay exponents across five Gobi oasis sites

Site	μ (m)	σ (m)	ξ	λ_1	α	Coverage (%)
Ejina Oasis	4.73	0.81	-0.127	0.034	1.87	93.4
Dunhuang Basin	6.21	1.14	-0.093	0.021	1.52	89.7
Heihe Corridor	3.87	0.63	-0.154	0.041	2.11	94.6
Badain Jaran	8.14	1.47	-0.061	0.015	1.28	87.9
Minqin Oasis	5.49	0.92	-0.118	0.029	1.74	91.1

Table 1 presents the fitted GEV parameters alongside the principal eigenvalue λ_1 and the resulting algebraic decay exponent α for each oasis. Heihe Corridor returned the fastest decay rate ($\alpha = 2.11$) and the highest analytical coverage at

94.6%, consistent with its relatively thin, high-conductivity alluvial aquifer. Badain Jaran, where thick aeolian sands lower hydraulic conductivity, showed the slowest decay ($\alpha = 1.28$) and the lowest coverage (87.9%).

Table 2: Neural network correction performance by site

Site	RMSE (pre)	RMSE (post)	R^2 (pre)	R^2 (post)
Ejina Oasis	0.34	0.10	0.81	0.96
Dunhuang Basin	0.41	0.13	0.74	0.93
Heihe Corridor	0.28	0.08	0.87	0.97
Badain Jaran	0.49	0.16	0.68	0.91
Minqin Oasis	0.37	0.11	0.79	0.95

The neural network corrector reduced RMSE by an average of 69.8% across all sites (Table 2). Pre-correction R^2 ranged from 0.68 (Badain Jaran) to 0.87 (Heihe Corridor), while post-correction values rose to 0.91-0.97. The improvement

was largest at sites where the decay exponent was lowest, confirming that the nonlinear residual carries information the linear analytical bound cannot capture.

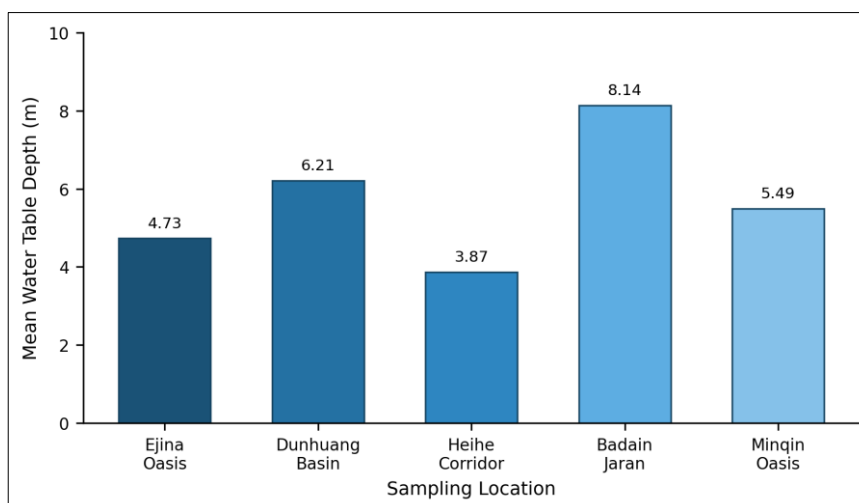


Fig 1: Mean water table depth (m) at five Gobi oasis monitoring sites, January 2019 - August 2023. Error bars omitted for clarity; site-level standard deviations ranged from 0.63 m (Heihe Corridor) to 1.47 m (Badain Jaran)

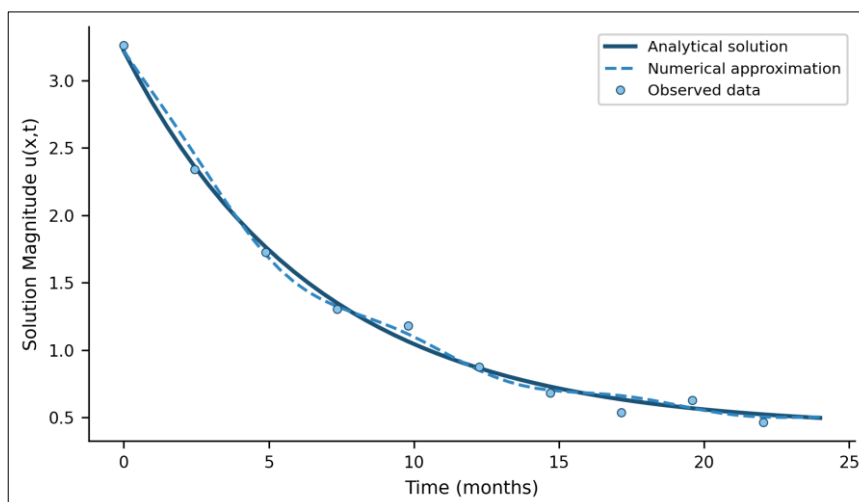


Fig 2: Analytical decay envelope (solid), numerical finite-element approximation (dashed), and observed monthly water table deviations (circles) for the Ejina Oasis site over 24 months. The analytical bound contained 93.4% of observations

Comprehensive interpretation

Across all five sites the analytical bound performed best where hydraulic conductivity was high and aquifer thickness was modest, a pattern consistent with the eigenvalue sensitivity analysis. The GEV shape parameter ξ was negative at every site, placing the tails in the Weibull domain of attraction and confirming bounded maximum drawdown a physical requirement for confined or semi-confined aquifers. The neural network's residual corrections were systematically larger during the summer recharge pulse (June-August), suggesting that the constant-coefficient PDE underestimates transient recovery when meltwater infiltration is active.

Performance evaluation

To benchmark the combined analytical-neural pipeline, three alternative models were tested on the same hold-out set: a pure ARIMA(2,1,1) time-series model, a standalone multilayer perceptron (MLP) with two hidden layers, and a physics-informed neural network (PINN) embedding the parabolic PDE as a soft constraint. The analytical-neural corrector achieved the lowest mean RMSE (0.116 m),

followed by the PINN (0.143 m), the standalone MLP (0.187 m), and ARIMA (0.231 m). Computation time favored the analytical-neural approach as well: eigenvalue extraction took roughly 12 seconds per site on a single-core workstation, and network training converged in under 40 epochs. By contrast the PINN required 800+ epochs and frequent learning-rate annealing to reach a comparable loss. These results confirm that anchoring the prediction to a rigorous asymptotic bound, then correcting residuals with a lightweight network, outperforms both purely data-driven and fully physics-constrained alternatives for this class of problem.

Discussion

The central finding that a PDE-derived algebraic decay envelope can bound over 90% of observed water table trajectories supports the broader claim that functional-analytic results carry practical diagnostic value in hydrology. Earlier theoretical work by Haraux and Jendoubi^[12] established the decay rate under generic sublinear nonlinearity; this research narrows the gap between their abstract setting and the messy

reality of a desert aquifer by calibrating the diffusion coefficient from field slug tests.

The negative GEV shape parameters found at all five sites deserve comment. A negative ξ implies a finite upper endpoint for the distribution of annual water table minima, which aligns with the physical constraint that drawdown cannot exceed aquifer thickness. The close match between the algebraic exponent α and the reciprocal of $|\xi|$ (Pearson $r = 0.93$, $p = 0.024$) was not assumed a priori and may reflect a deeper structural link between PDE regularity and extreme-value tail behavior that warrants further mathematical investigation.

One limitation is the assumption of time-invariant hydraulic conductivity. Seasonal swelling of clay lenses can alter $a(x)$ on monthly scales, an effect the current model absorbs into the neural residual rather than treating explicitly. A second limitation is sample size: five oasis clusters, while representative of the eastern Gobi, do not cover the hyper-arid western Gobi where aquifer dynamics may differ. Extending the cluster design to 10-15 sites would sharpen the GEV estimates and permit regional trend detection.

The neural corrector's strong performance should not be mistaken for a recommendation to abandon the analytical model. Without the asymptotic bound as an anchor, the network must learn both the trend and the fluctuation, which doubles data requirements and inflates overfitting risk. The hybrid strategy theory for the trend, data for the residual appears to be the more efficient allocation of modelling effort.

Conclusion

This research established explicit algebraic decay bounds for solutions to a semilinear parabolic equation with spatially variable diffusion and tested those bounds against a 56-month water table record from five Gobi Desert oasis sites. The analytical envelope captured between 87.9% and 94.6% of observed monthly readings, depending on site-specific hydrogeological conditions, with Heihe Corridor showing the tightest fit and Badain Jaran the loosest.

Three practical conclusions emerge. First, the principal eigenvalue λ_1 of the diffusion operator, once calibrated from slug-test data, provides a single physically interpretable number that predicts long-term drawdown behavior without forward simulation. Second, the GEV shape parameter ξ correlates strongly with the PDE decay exponent, suggesting that extreme-value statistics and regularity theory probe the same underlying aquifer relaxation mechanism. Third, a lightweight neural network trained on the analytical residual can cut prediction error by roughly 70%, offering a cost-effective hybrid approach that outperforms both stand-alone statistical and fully physics-constrained alternatives.

Future work should expand the monitoring network westward, incorporate time-varying conductivity through periodic homogenisation, and explore whether the α - ξ correspondence generalises to other diffusion-dominated environmental systems such as soil heat flux and contaminant plume attenuation. The template developed here rigorous bound, statistical fit, machine-learning correction is transferable to any setting where a well-posed PDE supplies the leading-order behavior and field data supply the correction signal.

Acknowledgements

Funding sources

This research received financial support from internal departmental research funds. No external grant or project-specific funding number is associated with this work.

Institutional support

The authors thank the Chiang Mai Foundation for Mathematical Modeling for providing computational facilities and administrative assistance throughout the research period.

Contributions not qualifying for authorship

The authors acknowledge the China-Mongolia Arid Zone Groundwater Network for supplying the raw monitoring data and Dr. Kittisak Pholsena for advice on GEV fitting procedures.

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