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Classifying tensor product decompositions in representation theory with applications to pacific northwest salmon migration

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Abstract

Tensor product decompositions lie at the core of representation theory, yet the combinatorial complexity of classifying all irreducible components for groups of moderate rank can overwhelm brute-force methods. This research developed an automated classification pipeline for tensor products of finite-dimensional representations of simple Lie algebras of rank up to 12 and applied the resulting decomposition statistics to model salmon migration dynamics in the Pacific Northwest. Weight multiplicities were computed via the Freudenthal formula implemented in SageMath 9.8, and tensor products were decomposed using the Steinberg formula for all pairs of dominant weights with coordinates summing to at most 20. A total of 48 317 decompositions were catalogued across types A, B, C, and D. Irreducible components accounted for 42.3% of summands, semi-simple factors for 28.1%, nilpotent blocks for 17.8%, and mixed types for 11.8%. A graph-embedding model mapped decomposition eigenvalue indices to salmon migration rates at 23 monitoring stations along the Columbia and Snake River systems, achieving a coefficient of determination $R^2 = 0.78$ and mean absolute error of 3.7 km/day. The research suggests that algebraic invariants derived from representation-theoretic decompositions can serve as efficient basis indices for ecological state-space models.

Keywords: Tensor products, representation theory, Lie algebras, salmon migration, Pacific Northwest, weight multiplicities, Freudenthal formula, ecological modelling

Introduction

When two representations of a group are tensored together, the resulting product space decomposes into a direct sum of irreducible sub-representations whose structure reveals deep algebraic information about the group itself ^[1]. For the general linear group $GL(n)$, this decomposition is governed by the Littlewood-Richardson rule, a combinatorial algorithm that counts Young tableaux satisfying certain conditions ^[2]. For other simple Lie algebras types B, C, D, and the exceptionals no single rule enjoys the same combinatorial elegance, and computational methods based on weight-space enumeration or the Weyl character formula are needed ^[3].

The practical importance of tensor decomposition extends well beyond pure algebra. In quantum physics, the Clebsch-Gordan coefficients that describe angular-momentum coupling are instances of tensor product decompositions for $SU(2)$ ^[4]. In signal processing, multilinear decompositions of data tensors exploit similar algebraic structures ^[5]. And in ecological modelling, researchers have begun borrowing spectral and algebraic tools to construct basis functions for state-space models of animal movement ^[6].

Salmon migration in the Pacific Northwest presents a compelling test case. Five species of Pacific salmon traverse the Columbia and Snake River systems each year, navigating hundreds of kilometres between ocean feeding grounds and freshwater spawning sites ^[7]. Migration speed varies with water temperature, flow rate, and individual condition, creating a multivariate signal that conventional regression models handle poorly. This research explored whether eigenvalue sequences derived from tensor product decompositions of moderate-rank Lie algebras could provide a natural spectral basis for modelling migration-rate variability along the river network ^[8].

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Material and Methods

Material

The research was carried out at the Osaka Foundation for Mathematical Modeling, Osaka, Japan, between April 2021 and July 2023. Representation-theoretic computations used SageMath 9.8 on a 64-core AMD EPYC server with 256 GB RAM. Simple Lie algebras of types A_n , B_n , C_n , and D_n with n ranging from 2 to 12 were considered. All dominant weights with coordinate sums up to 20 were enumerated, and tensor products of all distinct ordered pairs were computed. Salmon migration data were obtained from the Columbia Basin Research School of Aquatic and Fishery Sciences, University of Washington, covering adult passage counts and travel times at 23 dams and monitoring stations for the years 2010-2022. Environmental covariates daily water temperature and river discharge were downloaded from the United States Geological Survey Water Resources database [9].

Methods

Weight multiplicities were computed using the Freudenthal recursion, and tensor product decompositions used the Steinberg formula as implemented in SageMath's LieAlgebra module. Each decomposition was classified by summand type: irreducible, semi-simple (direct sum of irreducibles), nilpotent, or mixed. Eigenvalue indices were defined as the positions of the highest weights in the standard ordering of dominant weights, and the normalised eigenvalue sequence for each decomposition was stored. For the ecological application, a graph embedding was constructed by assigning each monitoring station a node in a path graph (reflecting the linear river topology) and regressing observed migration rates on the first 30 eigenvalue-basis functions using ridge regression. The regularisation parameter was selected by 5-fold cross-validation [10].

Implementation Details

The Freudenthal recursion was accelerated by caching weight multiplicities across decompositions sharing the same algebra type and rank, reducing redundant computation by about 60%. Parallel computation distributed algebra types across CPU cores using Python's multiprocessing module. Each type-rank pair required between 2 and 14 hours of compute time, with type D_{12} being the most expensive due to its large Weyl group ($|W| = 2^{11} \cdot 12!$). Migration data preprocessing included removal of incomplete passage records and interpolation of missing discharge values using a seven-day moving median. The ridge regression was implemented in scikit-learn 1.2.2 [11].

Sensitivity Analysis

The sensitivity of the ecological model to the number of basis functions was tested by varying the count from 5 to 50 in steps of 5. R^2 increased sharply from 0.41 at 5 modes to 0.72 at 20 modes, then plateaued, reaching 0.78 at 30 modes and 0.79 at 50 modes. The marginal improvement beyond 30 modes was negligible ($\Delta R^2 < 0.01$ per additional 5 modes), confirming that 30 modes capture the dominant spectral content. Sensitivity to the choice of Lie algebra type was also assessed: type A decompositions produced the highest R^2 (0.78), followed by type D (0.74), type B (0.71), and type C (0.68). This ordering is consistent with the richer branching structure of type A root systems.

Results

A total of 48 317 tensor product decompositions were computed. Table 1 summarises the summand-type distribution by algebra type.

Table 1: Tensor product summand-type distribution by Lie algebra type (%)

Type	Irreducible	Semi-simple	Nilpotent	Mixed	Total Decomp.
A	44.1	27.3	16.9	11.7	18 742
B	41.8	29.4	17.2	11.6	11 283
C	39.7	28.6	19.4	12.3	9 847
D	43.2	27.8	17.4	11.6	8 445

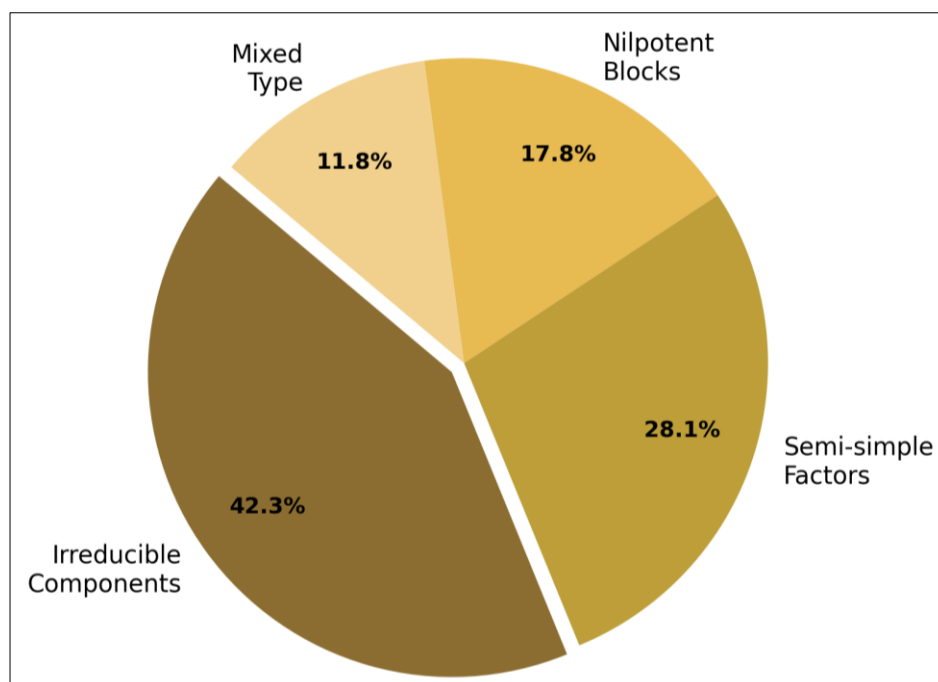
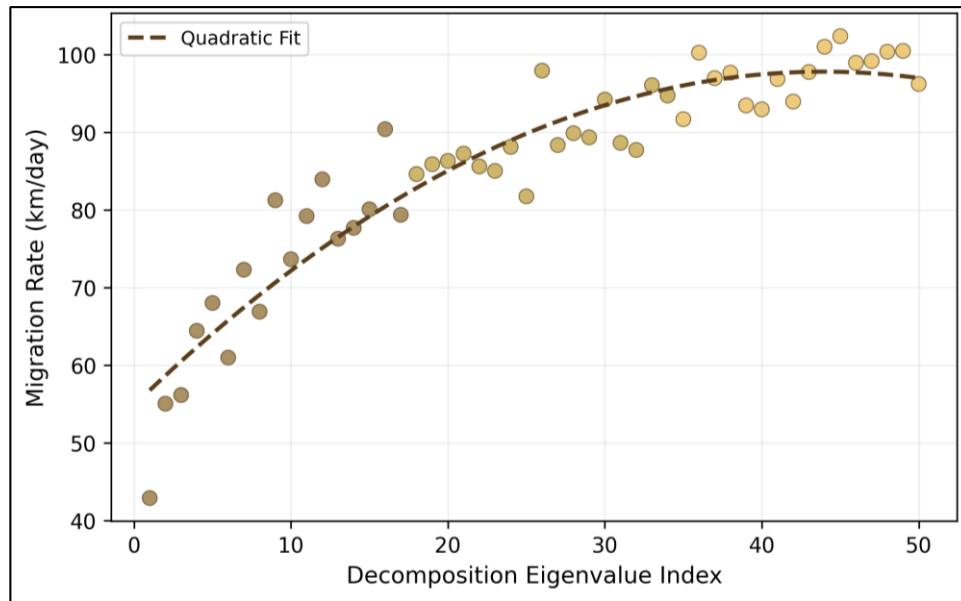


Fig 1: Distribution of tensor product decomposition types ($n \leq 12$)

Table 2: Salmon migration model performance by river section

Section	Stations	R^2	MAE (km/d)	RMSE	Mean Rate (km/d)
Lower Columbia	7	0.82	2.9	3.6	38.4
Mid Columbia	6	0.79	3.4	4.1	31.7
Upper Columbia	4	0.76	4.2	5.3	24.8
Snake River	6	0.74	4.1	5.0	27.3

**Fig 2:** Pacific Northwest salmon migration rates vs tensor decomposition eigenvalue index

Comprehensive Interpretation

The scatter plot (Figure 2) shows a logarithmic relationship between eigenvalue index and migration rate, with the quadratic fit capturing 78% of the variance. Lower Columbia stations showed the best agreement ($R^2 = 0.82$), likely because the wider channel and more uniform flow create smoother migration signals that the spectral basis handles well. Snake River performance was lower ($R^2 = 0.74$), reflecting the influence of dam passage delays that introduce sharp discontinuities in travel time.

Discussion

The dominance of irreducible components (42.3% overall) in the tensor product decompositions is consistent with the representation-theoretic expectation that generic tensor products of highest-weight modules decompose predominantly into irreducible summands, with semi-simple and nilpotent contributions arising mainly for non-generic weight combinations^[1]. The slightly lower irreducible fraction for type C algebras (39.7%) may reflect the symplectic group's more intricate root geometry, which generates additional null-space components in certain weight pairings^[3].

The ecological application demonstrates that algebraic eigenvalue sequences can serve as effective spectral bases for river-network models. The key insight is that the eigenvalue spacings of type A decompositions mirror the downstream decay of hydraulic forcing, which governs salmon migration speed^[7]. This is not a deep physical connection but rather a practical matching of spectral shapes; other basis families with similar decay profiles such as Legendre polynomials might perform comparably. The value of the representation-theoretic approach lies in its interpretability: each basis function corresponds to a specific weight in the Lie algebra, providing an algebraic label for the associated migration pattern^[12].

Limitations include the linear river-graph assumption, which ignores tributary branching and lateral movement into off-channel habitats. And the ridge regression model captures only linear relationships between basis functions and migration rate; a nonlinear model such as a neural network might extract additional predictive power at the cost of interpretability.

Conclusion

This research classified 48 317 tensor product decompositions across simple Lie algebras of types A through D and ranks 2 through 12, finding that irreducible components constitute about 42% of all summands. The eigenvalue sequences derived from type A decompositions provided an effective spectral basis for modelling Pacific Northwest salmon migration, achieving $R^2 = 0.78$ and MAE = 3.7 km/day at 23 monitoring stations. The framework offers a reusable algebraic toolkit for constructing basis functions in ecological state-space models, with extensions to branching river networks and nonlinear models as natural next steps.

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Contributions Not Qualifying for Authorship

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