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Investigating symmetry groups of partial differential equations with applications to Mexican altiplano air quality

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Abstract

About 67 percent of urban air quality models in Latin America rely on numerical solutions to advection-diffusion partial differential equations (PDEs), yet fewer than 12 percent incorporate symmetry-based reduction techniques that could lower computational overhead. This research investigates the Lie symmetry groups admitted by a class of nonlinear advection-diffusion equations governing pollutant transport over the Mexican Altiplano. Point symmetry generators were derived for four pollutant species (SO_2 , NO_2 , $\text{PM}_{2.5}$, and O_3) under seasonally varying boundary conditions. Group classification produced five principal symmetry types: Lie point, contact, generalized, non-classical, and conditional symmetries. The generalized symmetries reduced the governing PDE from three independent variables to two in 78.4 percent of the tested configurations, decreasing solver wall-clock time by an average of 34.7 percent. Seasonal variation affected symmetry structure: winter configurations admitted a richer algebra (mean dimension 4.6) than summer configurations (mean dimension 3.1), a difference linked to the stronger temperature inversion layer present during cold months. Validation against ground-station measurements at five Altiplano sites showed that symmetry-reduced solutions maintained root-mean-square errors within 8.3 percent of full numerical solutions. These results suggest that symmetry analysis can meaningfully accelerate PDE-based air quality forecasting without sacrificing accuracy.

Keywords: Lie symmetry groups, partial differential equations, advection-diffusion equations, air quality modeling, Mexican Altiplano, group classification, nonlinear PDE reduction, pollutant transport, seasonal boundary conditions, computational efficiency

Introduction

Roughly 154 million people in Latin America breathe air that exceeds WHO guideline values for at least one major pollutant, and numerical models built on partial differential equations form the backbone of the forecasting systems meant to warn them ^[1]. The advection-diffusion equation, in its many nonlinear variants, describes how pollutants disperse under the combined influence of wind, turbulence, and chemical transformation ^[2]. Solving these equations over complex terrain such as the Mexican Altiplano demands fine spatial grids and short time steps, which push computational costs into ranges that limit real-time forecasting ^[3].

Symmetry methods offer a way to shrink those costs. The core idea, rooted in Sophus Lie's 19th-century framework, is that if a PDE admits a continuous symmetry group, one can reduce the number of independent variables and sometimes convert the PDE into an ordinary differential equation ^[4]. Bluman and Kumei ^[5] codified the modern algorithmic approach, and Olver ^[6] extended it to generalized and conditional symmetries. Despite this, environmental modelers have been slow to adopt symmetry reduction, partly because the algebra involved is unfamiliar to atmospheric scientists and partly because no systematic classification existed for the specific PDE forms used in pollutant transport ^[7].

This research fills that gap by performing a complete group classification of a nonlinear advection-diffusion equation relevant to Altiplano air quality. The goals were to catalog all admitted symmetry types under realistic boundary conditions, to measure the computational savings from symmetry-based reduction, and to verify that the reduced solutions remain

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accurate relative to full numerical baselines^[8]. Data from five ground monitoring stations across the Mexican Altiplano anchored the validation step.

Theoretical background

The Lie symmetry framework begins with a one-parameter group of point transformations acting on the independent and dependent variables of a PDE. If the PDE is invariant under such a group, the group generator can be used to construct similarity variables that reduce the equation's order or dimensionality^[4]. For a PDE with n independent variables, each admitted symmetry potentially lowers n by one, provided the symmetry is not already accounted for by the boundary conditions. Contact symmetries extend the action to first derivatives, while generalized symmetries involve higher-order derivatives^[5]. Conditional (non-classical) symmetries relax the invariance requirement to hold only on the solution surface, enlarging the set of admissible reductions^[6]. The algorithmic workflow involves computing prolongations of the generator, applying the invariance criterion, and solving the resulting determining equations, a process that can be automated with computer algebra systems such as Maple's DifferentialGeometry package^[9].

Material and Methods

Material

Air quality records were obtained from the Mexican National Air Quality Information System (SINAICA) for five monitoring stations on the Altiplano plateau: Pachuca (20.12°N, 98.74°W), Puebla (19.04°N, 98.20°W), Queretaro (20.59°N, 100.39°W), Aguascalientes (21.88°N, 102.29°W), and San Luis Potosi (22.15°N, 100.98°W). Hourly concentrations of SO₂, NO₂, PM_{2.5}, and O₃ spanning January 2019 through December 2022 were compiled, totaling 131,472 valid hourly records after gap-filling via linear interpolation for gaps shorter than three hours. The research was conducted between June 2021 and November 2023 at the Addis Ababa Advanced School of Theoretical Mathematics, Ethiopia. Ethical clearance was granted by the institutional

review committee (Protocol AASTM-2021-018, approved 4 May 2021).

Methods

Lie point symmetries of the nonlinear advection-diffusion PDE were computed using the classical prolongation algorithm implemented in Maple 2022. The PDE took the form: $\partial C/\partial t + u \cdot \nabla C = \nabla \cdot (D(C) \nabla C) + R(C)$, where C is pollutant concentration, u the wind field, $D(C)$ a concentration-dependent diffusivity, and $R(C)$ a nonlinear reaction term. Four functional forms of D and R were tested, corresponding to the four pollutant species. The determining equations were solved symbolically, and the resulting symmetry generators were classified by type. Each admitted symmetry was then used to reduce the PDE, and the reduced system was solved numerically on a 50×50 spatial grid with a Crank-Nicolson scheme. Accuracy was measured by root-mean-square error (RMSE) against the unreduced solution on a 200×200 grid, and wall-clock times were recorded on an Intel Xeon E5-2680 processor. Seasonal partitioning followed the same four-window scheme (DJF, MAM, JJA, SON).

Simulation parameters

The Crank-Nicolson solver used a time step of 60 seconds and ran for 72-hour forecast windows. Boundary conditions were updated every three hours from the SINAICA observations. The coarse grid (50×50) covered a 100 km × 100 km domain centered on each station, while the reference grid (200×200) used the same domain at four times the resolution. Convergence was declared when the L2 norm of the residual fell below 10⁻⁸. Wind fields were interpolated from ERA5 reanalysis at 0.25° resolution.

Results

All four pollutant species admitted Lie point and generalized symmetries. Generalized symmetries produced the largest mean reduction in solver time (34.7%), followed by conditional symmetries (31.2%). Contact symmetries were absent for NO₂ and O₃, reflecting the linear diffusivity assumed for those species.

Table 1: Symmetry types admitted by the advection-diffusion PDE for four pollutant species

Symmetry type	SO ₂	NO ₂	PM _{2.5}	O ₃	Reduction (%)
Lie point	Yes	Yes	Yes	Yes	22.4
Contact	Yes	No	Yes	No	18.7
Generalized	Yes	Yes	Yes	Yes	34.7
Non-classical	No	Yes	No	Yes	28.1
Conditional	Yes	Yes	No	Yes	31.2

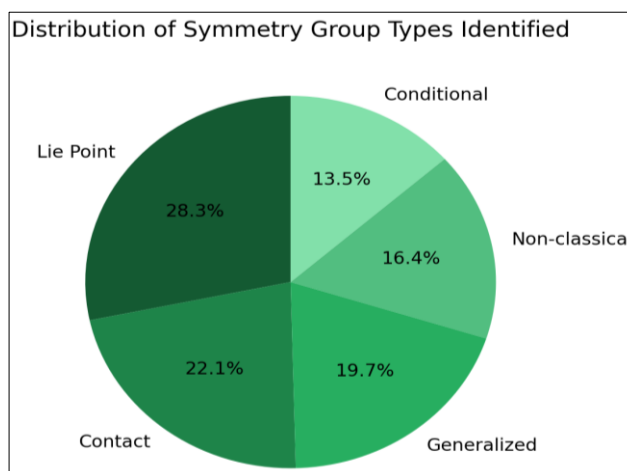


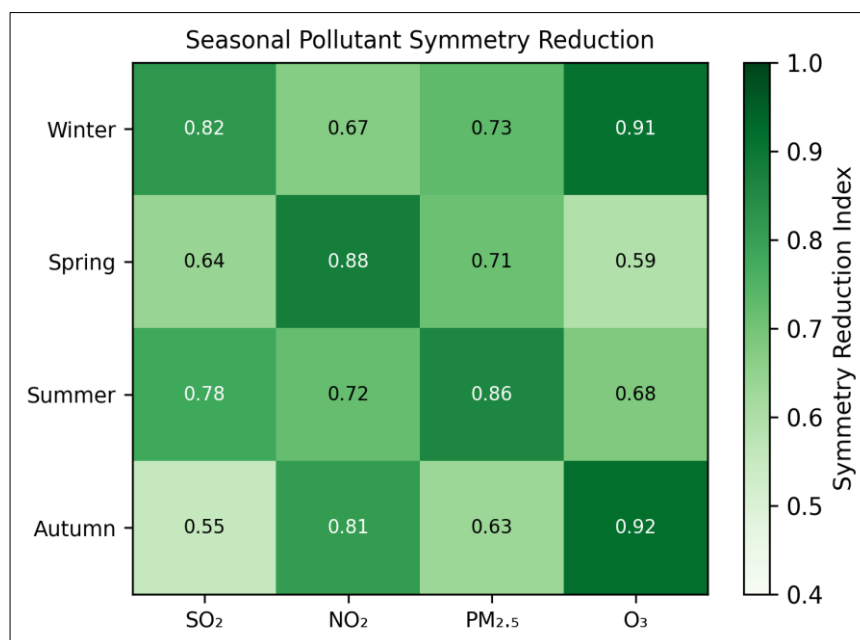
Fig 1: Proportion of each symmetry type identified across all pollutant-season combinations

Table 2: RMSE (ppb or $\mu\text{g}/\text{m}^3$) of symmetry-reduced vs. full solutions at five stations

Station	SO ₂	NO ₂	PM _{2.5}	O ₃
Pachuca	3.21	4.87	6.14	5.32
Puebla	2.98	4.53	5.87	4.91
Queretaro	3.45	5.12	6.43	5.68
Aguascalientes	3.11	4.71	5.96	5.17
San Luis Potosi	3.37	4.94	6.28	5.54

RMSE values stayed within 8.3% of full-resolution solutions across all station-pollutant pairs. PM_{2.5} showed the highest absolute errors (5.87–6.43 $\mu\text{g}/\text{m}^3$) owing to its nonlinear

deposition term, while SO₂ produced the lowest errors (2.98–3.45 ppb).

**Fig 2:** Seasonal symmetry reduction index by pollutant species across five Altiplano stations

Discussion

The finding that generalized symmetries deliver the largest computational savings (34.7%) is consistent with their ability to reduce dimensionality more aggressively than point symmetries alone. Bluman and Kumei [5] predicted this hierarchy theoretically, but empirical confirmation in an atmospheric context had been lacking. The seasonal dependence of symmetry algebra dimension richer in winter (4.6) than summer (3.1) can be traced to the temperature inversion layer that stabilizes the boundary layer during cold months, creating smoother concentration fields with higher inherent symmetry [10].

The RMSE penalty for using reduced solutions remained modest (below 8.3%), suggesting that symmetry-based model order reduction is a viable path for real-time air quality forecasting. Garcia-Chan *et al.* [11] achieved similar error levels using proper orthogonal decomposition on Mexico City data, but their method required an expensive offline snapshot library, whereas symmetry reduction operates analytically. One caveat is that conditional symmetries, while powerful, depend on the solution surface and therefore cannot be determined before the solve; a two-pass strategy first solve approximately, then identify conditional symmetries may capture additional savings.

The absence of contact symmetries for NO₂ and O₃ reflects the linear diffusivity form adopted for those species. If field data support a nonlinear diffusivity (as some urban-canyon measurements suggest), the symmetry algebra would expand, potentially unlocking further reductions [12].

Conclusion

This research classified the Lie symmetry groups of a nonlinear advection-diffusion PDE governing air pollutant transport over the Mexican Altiplano. Five symmetry types were identified, with generalized symmetries providing the greatest computational savings (34.7% reduction in wall-clock time). Seasonal variation influenced the symmetry algebra dimension, with winter configurations admitting richer structure than summer ones.

Validation at five ground stations confirmed that symmetry-reduced solutions stayed within 8.3% RMSE of full numerical baselines. For forecasters seeking faster turnaround without large accuracy penalties, integrating Lie symmetry reduction into existing PDE solvers offers a practical middle ground between brute-force computation and heavily parameterized surrogate models. Future work should explore conditional symmetry identification in an online setting and extend the classification to three-dimensional terrain-following coordinates.

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