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On robustness of a sequential probability ratio test for shape parameters of Lomax and Burr distributions under parameter variations

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Abstract

This paper investigates the robustness of the Sequential Probability Ratio Test (SPRT) for testing the shape parameter of the Lomax and Burr distributions. We focus on how variations in the scale parameter of the Lomax distribution and one of the shape parameters of the Burr distribution affect the performance of SPRT. Expressions for the operating characteristic (OC) and average sample number functions are derived for shape parameter of Lomax and Burr distributions and the OC functions are plotted for both distributions. The findings demonstrate the conditions under which SPRT remains efficient and reliable, providing valuable insights for practitioners working with these distributions.

Keywords: Pareto distributions, sequential probability ratio test, operating characteristic function, average sample number function, robustness

1. Introduction

The Lomax distribution also known the Pareto Type II distribution is a heavy tailed distribution originally proposed by Lomax in 1987. Lomax distribution is widely used in the discipline of Statistics, especially in life-testing, survival analysis, actuarial science, economics, business and reliability analysis. The distribution, which is basically a shifted Pareto distribution. Lomax distribution has scale parameter λ and shape parameter θ . The Lomax distribution It is utilised to effectuate realistic modifications to the Pareto distribution in life-testing scenarios. The Lomax distribution is a particular instance of the Burr distribution.

Burr distribution is widely used in survival analysis and study of income distribution (Refer Singh and Maddala (1976) [7]. It has two shape parameters θ and c . Burr distribution has the hazard rate function which can be either decreasing or upside-down bathtub-shaped. It has a tail index like Pareto distribution. This fact implies that Burr distribution is heavy-tailed. Therefore, it can be an alternative model for a heavy-tailed survival time data. Burr distribution can be used to model the survival time whose hazard rate function decreases. It can be used in the various fields of sciences, including reliability analysis, life testing, survival analysis, actuaries, economics, forestry, hydrology, and meteorology.

Many authors have investigated the robustness of the sequential probability ratio test (SPRT) across various probability distributions amid alterations in the distribution. Researchers have considered the problem of sequential probability ratio tests of life distributions- to mention a few- Surinder and Chandra (2009) [8], Surinder Kumar and Mukesh Kumar (2013) [9], Wald, (1947) [10], to mention few. Barlow and Proschan (1967) [1], Harter and Moore (1976) [4], Montagne and Singapurwalla (1985), Chaturvedi et.al., (1998) [3], Chaturvedi et.al. (2002) [2], Pandit and Gudaganavar (2010) [6], among other researchers.

In this study, we investigate the robustness of SPRT for the problem of testing a simple hypothesis versus a simple alternative about the shape parameter, assuming the other parameter is known when the underlying distributions of sequence of random variables are

Lomax distribution and Burr distribution as given above.

The paper is organized as below:

Section 2 deals with the problem of constructing Sequential Probability Ratio Test (SPRT) for evaluating the hypothesis concerning the shape parameter by deriving equations for the Operating Characteristic (OC) and Average Sample Number (ASN) functions. Section 3 examines the robustness of the SPRT concerning OC functions when the distribution experiences a change in the scale parameter. In Section 4, we examined the robustness of the SPRT concerning the shape parameter of the Burr distribution in relation to the change in the other shape parameter.

2. Methodology

2.1 SPRT for Shape Parameter of Lomax Distribution

Let $\mathbf{X} = (X_1, X_2, \dots)$ be an infinite sequence of independent and identically random variables from a Lomax distribution with shape parameter $\theta (> 0)$ and known scale parameter $\lambda (> 0)$ with probability density function is

$$f(x; \theta, \lambda) = \frac{\theta \lambda^\theta}{(x + \lambda)^{\theta+1}}, x \geq 0 \quad (1)$$

In the following, SPRT is developed for the problem of testing simple null against simple alternative hypothesis of shape parameter in Lomax distribution.

$$\text{Let } Z = \ln \left(\frac{f(x; \theta_1)}{f(x; \theta_0)} \right) = \ln \left(\frac{\theta_1}{\theta_0} \right) + (\theta_1 - \theta_0) \ln \lambda - (\theta_1 - \theta_0) \ln (x + \lambda) \quad (2)$$

Let (α, β) be the desired strength of SPRT, then according Wald ^[4], A and B are approximately given by $A \approx \frac{1-\beta}{\alpha}$, $B \approx \frac{\beta}{1-\alpha}$, where α and β are both within the interval (0,1).

The Operating Characteristic (OC) function $L(\theta)$ is given by

$$L(\theta) = \frac{A^{h(\theta)} - 1}{B^{h(\theta)} - A^{h(\theta)}} \quad (3)$$

Where $h(\theta)$ is the non-zero solution of

$$E(e^{h(\theta)Z}) = 1 \quad (4)$$

From (1) and (2), we have

$$E(e^{h(\theta)Z}) = \int_0^\infty \left(\frac{\theta_1}{\theta_0} \right)^{h(\theta)} \lambda^{h(\theta)(\theta_1 - \theta_0)} (x + \lambda)^{-h(\theta)(\theta_1 - \theta_0)} \frac{\theta \lambda^\theta}{(x + \lambda)^{\theta+1}} dx$$

From (4), we get

$$\theta = \frac{h(\theta)(\theta_1 - \theta_0)}{\left(\frac{\theta_1}{\theta_0} \right)^{h(\theta)} - 1} \quad (5)$$

The Average Sample Number (ASN) function is given by

$$E(N|\theta) = \frac{L(\theta) \ln B + [1 - L(\theta)] \ln A}{E(Z)} \quad (6)$$

Provided $E(Z) \neq 0$,

Where,

$$E(Z) = \ln \left(\frac{\theta_1}{\theta_0} \right) - \frac{(\theta_1 - \theta_0)}{\theta} \quad (7)$$

The maximum value of ASN occurs in the neighbourhood of $\theta = \tilde{\theta}$, say where $\tilde{\theta}$ is the solution of $E_{\tilde{\theta}}(Z) = 0$ and this value is given by

$$E_{\tilde{\theta}}(Z) = \frac{-\ln A \ln B}{E_{\tilde{\theta}}(Z^2)} \quad (8)$$

It is easy to see that

$$\tilde{\theta} = \frac{(\theta_1 - \theta_0)}{\ln \left(\frac{\theta_1}{\theta_0} \right)} \quad (9)$$

And

$$E_{\tilde{\theta}}(Z^2) = \left(\ln \left(\frac{\theta_1}{\theta_0} \right) \right)^2 \quad (10)$$

Table 1 contains the values of $E_0(N)$, $E_1(N)$ and $E_{\bar{\theta}}(N)$ for different values of α , β and ratios of θ_0 and θ_1 .

Table 1: ASN Values of SPRT for scale parameter of Lomax distribution.

$\lambda = 5, \alpha = 0.01, \beta = 0.01$				$\lambda = 5, \alpha = 0.05, \beta = 0.05$			
θ_1/θ_0	$E_0(N)$	$E_1(N)$	$E_{\theta}(N)$	θ_1/θ_0	$E_0(N)$	$E_1(N)$	$E_{\theta}(N)$
0.1	3.2107	0.6724	3.9826	0.1	1.8894	0.3957	1.6352
0.2	5.5634	1.8837	8.1516	0.2	3.2739	1.1085	3.3470
0.3	8.9354	3.9874	14.5667	0.3	5.2582	2.3465	5.9810
0.4	14.2376	7.7148	25.1494	0.4	8.3784	4.5399	10.3262
0.5	23.3150	14.6755	43.9484	0.5	13.7201	8.6360	18.0449
0.6	40.6334	28.8962	80.9186	0.6	23.9114	17.0045	33.2246
0.7	79.4569	62.6347	165.9772	0.7	46.7578	36.8585	68.1491
0.8	194.5776	167.6773	424.0578	0.8	114.5025	98.6726	174.1151
0.9	840.0717	783.0872	1902.1200	0.9	494.3545	460.8210	780.9970

$\lambda = 6, \alpha = 0.01, \beta = 0.01$				$\lambda = 6, \alpha = 0.05, \beta = 0.05$			
θ_1/θ_0	$E_0(N)$	$E_1(N)$	$E_{\theta}(N)$	θ_1/θ_0	$E_0(N)$	$E_1(N)$	$E_{\theta}(N)$
0.1	3.2107	0.6724	3.9826	0.1	1.8894	0.3957	1.6352
0.2	5.5634	1.8837	8.1516	0.2	3.2739	1.1085	3.3470
0.3	8.9354	3.9874	14.5667	0.3	5.2582	2.3465	5.9810
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Table 2: ASN Function for $\lambda < \lambda^*$ $H_0: \theta = 5$ against $H_1: \theta = 10$ $\alpha = \beta = 0.05$

$\lambda^*\theta$	5.00	6	7	8	9	10
1	0.635703	0.575017	0.530470	0.495239	0.467232	0.444140
2	1.493468	1.274324	1.124022	1.013919	0.929470	0.862378
3	2.748738	2.211324	1.872299	1.637795	1.465815	1.333578
4	4.778457	3.556598	2.869494	2.427767	2.119068	1.890661
5	8.636041	5.667657	4.279109	3.427792	2.943789	2.561282
6	18.84416	9.474497	6.435007	4.927541	4.025397	3.424023
7	124.66	18.41765	10.15509	7.103086	5.512112	4.534399
8	136.85	64.59573	18.12899	10.7182	7.689673	6.041544

2.2 Robustness of the SPRT for shape parameter of Lomax distribution

In this section, the possibility of a misspecification for of the scale parameter λ and its effect on testing for the shape parameter using the SPRT in the Lomax distribution is analysed.

Then the probability density function of (1) becomes $f(x; \theta, \lambda^*)$. To study the robustness of the SPRT developed in Section 2 with respect to OC function, consider $h(\theta)$ as the solution of the equation:

$$E(e^{h(\theta)z}) = 1$$

$$\text{i.e., } \int_0^\infty \left(\frac{f(x; \theta_1, \lambda)}{f(x; \theta_0, \lambda)} \right)^{h(\theta)} f(x; \theta, \lambda^*) dx = 1$$

Giving

$$\left(\frac{\theta_1}{\theta_0} \right)^{h(\theta)} \lambda^{h(\theta)(\theta_1 - \theta_0)} \cdot \theta \lambda^{*\theta} \int_0^\infty \frac{(x + \lambda)^{-h(\theta)(\theta_1 - \theta_0)}}{(x + \lambda^*)^{\theta+1}} dx = 1 \quad (11)$$

We can solve above equation for different values of $\theta, h(\theta)$ [Using Mathematica and R Software] and the corresponding OC function is obtained. The robustness of SPRT with respect to ASN function can be studied by replacing the denominator of (6) by

$$E_{\lambda^*}(Z) = \int_0^\infty z f(x; \theta, \lambda^*) dx$$

$$E_{\lambda^*}(Z) = \ln \left(\frac{\theta_1}{\theta_0} \right) + (\theta_1 - \theta_0) \ln \lambda - (\theta_1 - \theta_0) \theta \lambda^{*\theta} \int_0^\infty \frac{\ln(x + \lambda)}{(x + \lambda^*)^{\theta+1}} dx \quad (12)$$

We consider the cases $\lambda < \lambda^*$ and $\lambda > \lambda^*$ to study the robustness of the SPRT. In Table 2, we present the ASN function for the cases $\lambda < \lambda^*$ for the SPRT of testing the hypothesis $H_0: \theta = 5$ against $H_1: \theta = 10$. The values of OC function for the cases $\lambda < \lambda^*$ and $\lambda > \lambda^*$ respectively are plotted in Figure 1 and Figure 2.

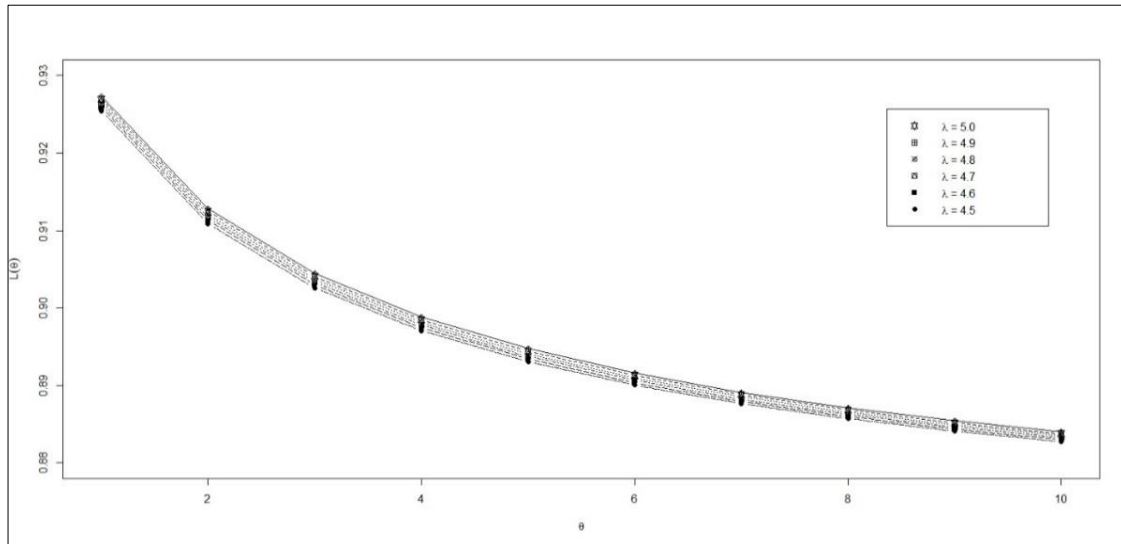


Fig 1: OC function decreases as parameter values increase below λ^*

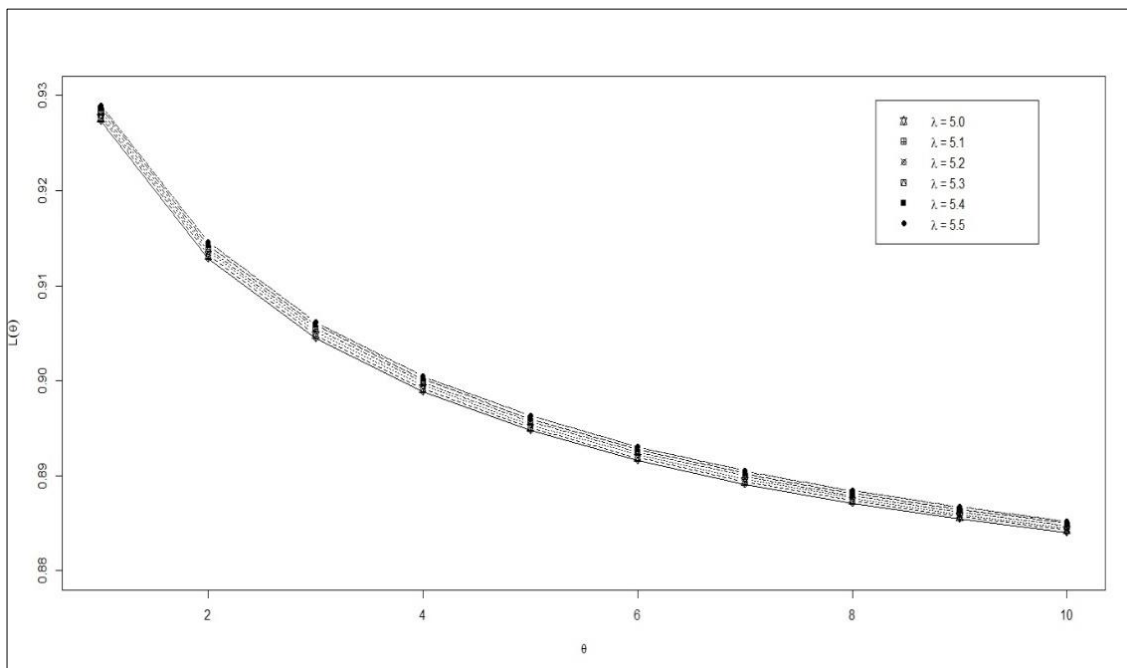


Fig 2: OC function behavior demonstrates robustness for parameter values above λ^*

2.3 SPRT for shape parameter of Burr distribution

In this sub section, SPRT for the testing for the shape parameter of Burr distribution. The random variable X is said to follow Burr distribution with shape parameters θ and c , if its probability density function is given by

$$f(x; \theta, c) = \frac{c\theta x^{c-1}}{(1+x^c)^{\theta+1}}, x > 0, \theta > 0, c > 0 \quad (13)$$

Where the parameter c is assumed to be known.

Given a sequence of observations $X_1, X_2, \dots$, from (13), we wish to test the null hypothesis $H_0: \theta = \theta_0$ against the alternative $H_1: \theta = \theta_1 (> \theta_0)$. We propose the following SPRT.

$$\text{Let } Z = \ln \left(\frac{f(x; \theta_1)}{f(x; \theta_0)} \right) = \ln \left(\left(\frac{\theta_1}{\theta_0} \right) (1 + x^c)^{-(\theta_1 - \theta_0)} \right) \quad (14)$$

The OC function $L(\theta)$ is given by

$$L(\theta) = \frac{A^{h(\theta)} - 1}{B^{h(\theta)} - A^{h(\theta)}} \quad (15)$$

Where $h(\theta)$ is the solution of $E(e^{h(\theta)Z}) = 1$.

Here, $E(e^{h(\theta)z}) = \int_0^\infty \left(\frac{\theta_1}{\theta_0}\right)^{h(\theta)} (1+x^c)^{-(\theta_1-\theta_0)} \frac{c\theta x^{c-1}}{(1+x^c)^{\theta+1}} dx = 1$

Hence, we obtain,

$$\theta = \frac{1-h(\theta)[(\theta_1-\theta_0)+1]}{\left(\frac{\theta_1}{\theta_0}\right)^{h(\theta)} - 1}.$$

2.4 Robustness of the SPRT for shape parameter of Burr distribution

The present subsection is concentrated on robustness of the SPRT for testing the hypothesis about shape parameter. Let us suppose that the other shape parameter undergone change and the probability density function (13) becomes $f(x; \theta, c^*)$. To study the robustness of the SPRT developed here with respect to OC function, we consider $h(\theta)$ as the solution of the equation

$$\int_0^\infty \left(\frac{f(x; \theta_1, c)}{f(x; \theta_0, c)}\right)^{h(\theta)} f(x; \theta, c^*) = 1$$

Giving

$$\left(\frac{\theta_1}{\theta_0}\right)^{h(\theta)} c^* \theta \int_0^\infty (1+x^c)^{-(\theta_1-\theta_0)} \frac{x^{c^*-1}}{(1+x^{c^*})^{\theta+1}} dx = 1 \quad (16)$$

One can solve above equation for different values of $\theta, h(\theta)$ [Using Mathematica and R Software] and OC function can be obtained.

The value of OC function for the cases $c > c^*$ and $c < c^*$ respectively are plotted Figure 3 and Figure 4.

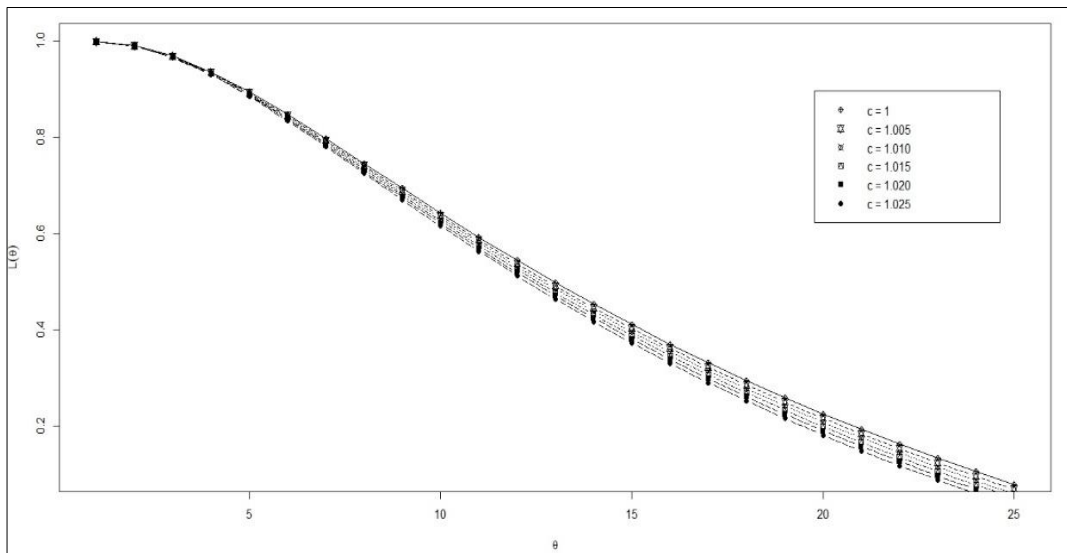


Fig 3: OC function values plotted for cases where c exceeds c

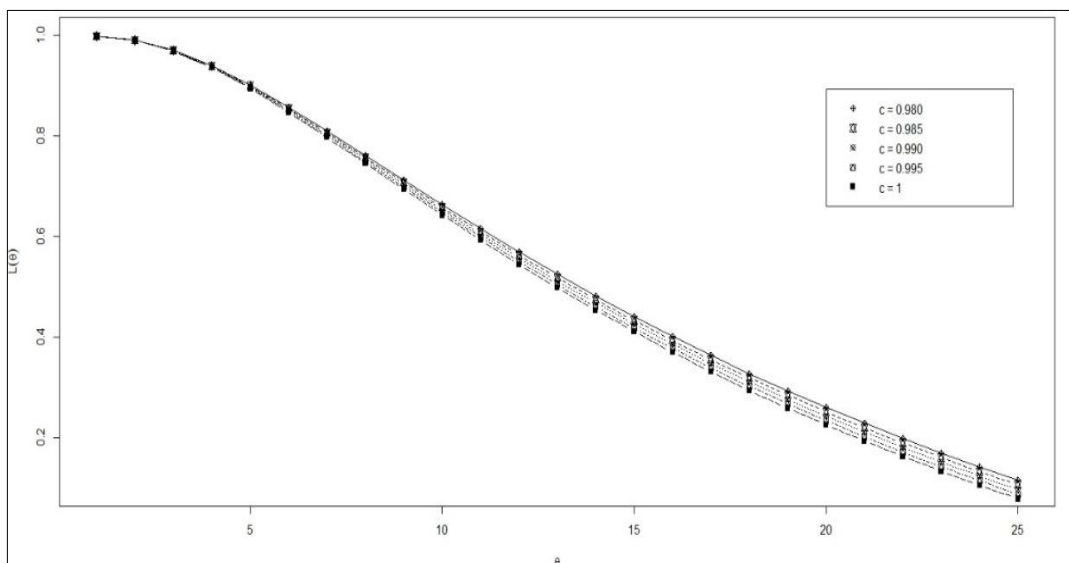


Fig 4: OC function values plotted for cases where c remains below c*

3. Conclusion

- The values of the OC function for the scenarios where $\lambda < \lambda^*$ and $\lambda > \lambda^*$ are illustrated in Figure 1 and Figure 2, respectively. The figures demonstrate that when $\lambda < \lambda^*$, the OC curves shift to the right, whereas for $\lambda > \lambda^*$ the OC curves shift to the left of the curve corresponding to $\lambda = \lambda^*$. It is evident from the figures that the SPRT is robust for $\lambda^* = \lambda \pm 0.1$, as the variation in the OC function is minimal.
- We have analysed the SPRT for evaluating the null hypothesis $H_0: \theta = 5$ against the alternative $H_1: \theta = 10$ for $\alpha = \beta = 0.05$. The operating characteristic (OC) function values for scenarios where c exceeds c^* and where c is less than c^* are illustrated in Figures 3 and 4, respectively. The figures demonstrate that when $c > c^*$, the OC curves shift to the right, whereas for the case $c < c^*$ OC curves shift to the left of the curve corresponding to $c = c^*$. It is evident from the figures that the SPRT maintains its robustness for $c^* = c \pm 0.005$, as the variations in the OC function are minimal.

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