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An inventory model that incorporates a three-echelon supply chain for product deterioration over a finite planning horizon

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Abstract

In this research study, we aim to formulate a coordinated three-echelon supply chain inventory mathematical framework that accounts for deterioration and time-dependent demand. We develop our model under a specified planning duration. The proposed framework is developed under a no-shortage condition, and replenishment quantity is instantaneous rather than fixed. We solved our model for both cases, centralized and decentralized. The essential purpose of this research article is to identify the optimum inventory cycle length for each order. This article helps all supply chain participants make collective decisions to minimize total inventory cost, which comprises ordering, purchasing, inventory, and deterioration costs. In the present study, several numerical illustrations are included that help to suggest a method and its optimal results. We also explained the impact that is made to interchange some factors. This is expressed in terms of a table as well as some graphs and sensitivity analysis using Mathematica software. These inspections are implemented to optimize the aggregate inventory-related cost of deteriorating products. These outcomes are particularly beneficial for the management of any organisation and for various business enterprises.

Keywords: Three-echelon supply chain, finite planning horizon, product deterioration

Introduction

Throughout the Second World War, operations research (OR) began to optimize the utilization of military resources. After that, it expanded to business and industry. It also helps improve decision-making power and efficiency. OR utilizes mathematical models and scientific methods to solve complex real-world problems. A supply chain is composed of a wide range of individuals, organizations, resources, and technologies deeply engaged in designing, manufacturing, and selling a product. It encompasses the complete sequence of activities, beginning with material sourcing to distributing the finished product to end-users. It also makes the connection among manufacturers, distributors, suppliers, and retailers. In the supply chain process, raw materials are processed into essential components that are later used to create final products. Moreover, customer service contributes significantly to supporting product delivery, returns, and replacements. In the supply chain, there are numerous participants, including manufacturers, transportation providers, suppliers, distribution centers, and retailers as well. The production and distribution system is not a single entity, but rather a connected series of organizations. According to Lee and Billington (1993) ^[1], the distribution chain is a network of facilities, including vendors, producers, and distributors, that converts raw materials into partially/fully finished products and delivers these products to customers. Integrated supply management refers to the process of managing all operations involved in manufacturing and delivering products and services to final customers. Effective production and distribution management contributes to the smooth movement of materials, data, and financial funds among suppliers, producers, distributors, retailers, and customers. SCM manages three primary flows:

- **Materials:** It includes the movement of raw material, work-in-process, and finished goods.

- **Information:** It includes the transmission of data like orders, inventory levels, and customer demand throughout the network of the supply chain.
- **Finances:** It includes managing the payments and financial arrangements between partners.

Managing this logistical network of the supply chain is a responsible duty at every step. If handled appropriately, it will reduce operational costs and enhance profitability. On the contrary, poor supply chain performance may cause inefficiencies, delays, and financial losses. Management of time in any firm, industry, or organisation plays a very crucial role. A finite horizon planner must develop a strategic plan with a predetermined timeframe. In terms of inventory management, it refers to the time period or duration during which the level of inventory is managed. This horizon can be either finite or infinite. It is totally dependent upon the participation of organisations and demand factors. Although the primary objective of this research is to focus especially on a specified planning duration, and derive the optimal cycle length for minimizing total inventory operation costs. Managing inventory of deteriorating (perishable) items across multiple echelons (supplier → manufacturer → distributor → retailer) is challenging because items lose value/quality over time, demand can be time- and stock-dependent, and coordination across levels dramatically affects total cost and service. Finite planning horizons (such as seasonal products, project-based supply, and promotional periods) further complicate optimization because replenishment cycles do not need to repeat. Present money has greater purchasing and earning power than future money. Deterioration generally refers to decay, in which any goods and products become unusable. It refers to the term in which the product becomes damaged, broken, spoiled, or expired. It refers to the loss of quantity over time. The rate of deterioration has become a huge issue. Inventory deterioration is a common occurrence in our daily lives. Various types of products are affected by deterioration. The first type of deteriorating items is destroyed over time, such as vegetables, flowers, and fruits. The second category of deteriorating items refers to those goods that reduce their value due to technological changes, such as computers and mobile phones. In finite planning horizons, the central objective is to minimize the overall cost within a specific time period.

Literature review

Over the past few years, a significant number of researchers has been conducted on inventory models for deteriorating items. Generally, deterioration is described as the process of decay, change, or spoilage that leads to loss of usability of the item for its intended function. As a result, controlling and maintaining the stock of deteriorated items is a computationally intensive decision for decision-makers. Despite extensive research on deteriorating item inventory systems, the influence of investment in reducing product deterioration remains largely unexplored. The present study formulated an inventory model to account for the implementation of preservation technology that reduces the rate of deterioration, enabling the retailer to minimise economic losses, enhance service performance levels, and boost competitive position in the market. Demand forecasting is a crucial element in shaping business strategies, providing organisations with the ability to optimise operations, reduce costs, and meet consumer expectations. The earliest study involving deteriorating items was pioneered by [2], who analyzed the decay of fashion products after the specified storage duration. Dye *et al.* [3] investigated an optimal pricing and order quantity considering a variable deterioration backorder mechanism. A model based on the stock system is formulated by Mishra and Singh [4], who established an inventory framework with deterioration effects, which is based on time -dependent demand and also considering the cost of holding, including a limited backordering policy. Despite its importance, little attention has been paid in recent years to the use of preservation technology (PT) and reducing deterioration has received limited attention in the literature. The adoption of PT has become important due to changing social conditions and its capability to considerably lower the deterioration rate. Efforts toward preservation technology investment help to reduce the rate of deterioration effectively. Mahapatra *et al.* [5] showed the benefit of preservation technology under EOQ-based inventory systems for time-dependent deterioration under the fuzzy environment. Chang *et al.* [6] developed an optimal inventory replenishment model for products experiencing non-instantaneous deterioration and stock-dependent demand. Goyal and Giri [7] analyzed a review of recent advances in deteriorating inventory modelling and categorized them according to demand fluctuations and other relevant conditions or constraints. Yang [8] conducted a comparative analysis of various lot-sizing models with partial backordering for product deterioration using maximum profit as the comparison criterion.

Pal *et al.* [9] established a multi-echelon supply chain framework for reprocessible products across multiple market environments. In their model, two suppliers were responsible for supplying raw materials to the manufacturer. The first supplier is considered the principal supplier, which is subject to supply disruption risks at a random time, whereas the second supplier serves as a reliable alternative supplier; however, it charges a higher expense than the first supplier. Two economic production quantity frameworks for deteriorating products with uniform and exponential distributions affected by random machine failures and fast corrective measures by Widvadana and Wee [10]. Skouri *et al.* [11] proposed an inventory system incorporating ramp-type demand, partial shortages, and Weibull deterioration behaviour. Chang *et al.* [12] initially introduced an inventory framework with a varying deterioration rate and proportional time-varying demand, incorporating trade credit arrangements. Jaggi *et al.* [13] formulated a coordinated supply chain system involving a supplier and retailer for obtaining optimal inventory policies under centralized control, a supplier-led Stackelberg strategy and Nash equilibrium approaches. Hung [14] established an inventory framework considering generalized demand patterns deterioration, and shortage backordering rates. Mishra and Jain [15] look forward to a decentralized supply chain optimisation model that incorporates blockchain and uses an iterative strategy to calculate total costs for retailers and suppliers.

All such researchers established their inventory frameworks by assuming that the customer demand rate is constant, time-varying, or inventory-dependent. Despite their usefulness, such assumptions are quite restrictive in representing actual situations accurately. For some commodities in practical situations, demand for certain products, particularly best-selling consumer goods, is affected by both time and stock availability. In addition, deterioration of items is a common phenomenon for particular consumer commodities. Motivated by these observations, this research aims to design an inventory framework for deteriorating products

with stock-dependent and linearly increasing demand, which will fit a more general inventory feature. This is the paper's major contribution.

Assumptions and Notations

In the present research, we adopted several key assumptions and incorporated additional assumptions and parameters wherever necessary.

Here, the demand pattern is described by the following expression: $D(t) = a + bt$.

In this equation, demand is time-dependent (t). The parameter a refers to the initial or maximum potential market demand when prices have no negative influence. The coefficient b indicates the demand that depends on time and decreases over time.

1. The time planning horizon is finite.
2. Lead time is assumed to be zero.
3. The model does not permit any shortages.
4. Replenishment quantity is instantaneous rather than fixed.

Notations

Furthermore, the following parameters and symbols are considered in the formulation of the proposed model:

1. IC - Inventory carrying cost per unit.
2. DC - Deterioration cost per unit.
3. CC- Capital cost of an item (₹ per unit per year).
4. WP - Wholesale price per unit.
5. SC - Set-up cost per replenishment cycle.
6. OC - Ordering cost incurred per order.
7. PC - Purchasing cost of an item (₹ per unit per year).
8. HC - cost of holding an item (₹ per unit per year).
9. T - Total planning horizon.
10. T_j - Length of each replenishment cycle.
11. TCS_{NC} - Supplier's total cost without coordination over the predetermined planning horizon T.
12. TCS_{WC} - Supplier's total cost with coordination over the predetermined planning horizon T.
13. TCR_{NC} - Total cost of the retailer without coordination over the predetermined planning horizon T.
14. TCR_{WC} - Total cost of the retailer with coordination over the predetermined planning horizon T.
15. t_i - Time during the i^{th} replenishment cycle when inventory reaches zero. θ
16. $i = 0, 1, 2, \dots, n_1 - 1$
17. t_j - Time during the j^{th} replenishment cycle when inventory reaches zero.
18. $j = 0, 1, 2, \dots, n_2 - 1$.
19. $I_{i+1}(t)$ - The level of inventory for $(i + 1)^{th}$ cycle at time t . When coordination is absent.
20. $t_i \leq t \leq t_{i+1}$
21. $I_{j+1}(t)$ - The level of inventory for $(j + 1)^{th}$ cycle at time t . When coordination is absent.
22. $t_j \leq t \leq t_{j+1}$
23. Q_{i+1} - Order quantity for the $(i + 1)^{th}$ cycle without coordination, placed at time t_i ,
24. where $i = 0, 1, 2, \dots, n_1 - 1$.
25. Q_{j+1} - Order quantity for the $(j + 1)^{th}$ cycle with coordination, placed at time t_j ,
26. where $j = 0, 1, 2, \dots, n_2 - 1$.
27. θ = Deterioration rate

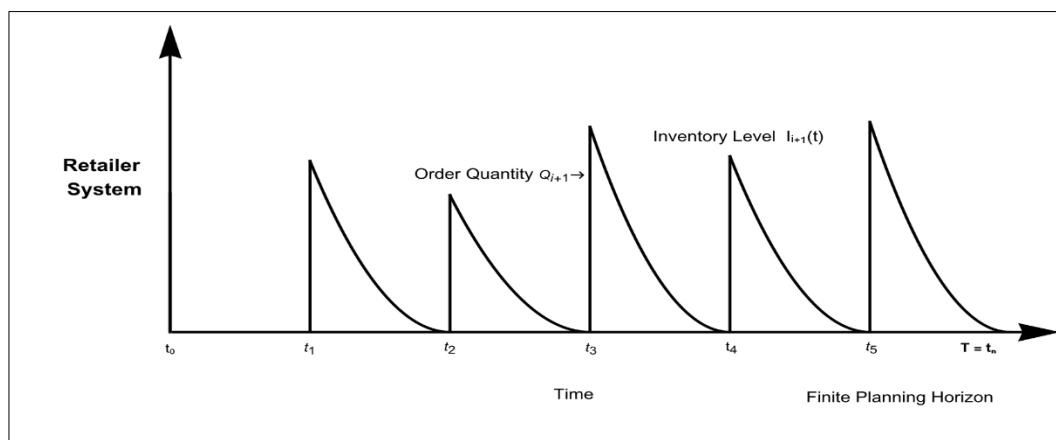


Fig 1: Inventory Variation Graph

Mathematical Formulation of the Constructed Framework and Its Solution Approach

Framework for Decentralized Strategic Planning Across a Bounded Planning Horizon

Assume that T is the total planning duration that is partitioned into n equal sub-intervals, with the length of each segment $T_j = T/n$. At the starting point, when $t_0 = 0$, the initial replenishment batch of size Q_{j+1} has already been consumed. As market demand

intensifies, we observe a noticeable decline in the inventory levels during each interval $[0, t_i]$. This demand can be expressed as $D(t) = a + bt$. Therefore, the level of stock is represented by using a mathematical differential equation is:

$$\frac{dI_{i+1}(t)}{dt} = -D(t) - \theta I_{i+1}(t) \quad t_i \leq t \leq t_{i+1} \quad \text{Eqn (1)}$$

$$\frac{dI_{i+1}(t)}{dt} + \theta I_{i+1}(t) = -D(t)$$

Multiply by $e^{\theta t}$

$$\left(\frac{dI_{i+1}(t)}{dt} + \theta I_{i+1}(t) \right) e^{\theta t} = -D(t) e^{\theta t} \quad \text{Eqn (2)}$$

Integrate t to t_{i+1}

$$I_{i+1}(t_{i+1}) e^{\theta t_{i+1}} - I_{i+1}(t_i) e^{\theta t_i} = - \int_{t_i}^{t_{i+1}} D(u) e^{\theta u} du \quad \text{Eqn (3)}$$

Boundary conditions

$$I_{i+1}(t_{i+1}) = 0 \quad I_{i+1}(t_i) = Q_{i+1} \quad \text{Eqn (4)}$$

$$I_{i+1}(t) = e^{-\theta t} \int_{t_i}^{t_{i+1}} D(u) e^{\theta u} du \quad \text{Eqn (5)}$$

$$I_{i+1}(t) = e^{-\theta t} \int_{t_i}^{t_{i+1}} (a + bu) e^{\theta u} du$$

$$I_{i+1}(t) = e^{-\theta t} \int_{t_i}^{t_{i+1}} (ae^{\theta u} + bue^{\theta u}) du$$

$$I_{i+1}(t) = \frac{(a+bt_{i+1})e^{\theta(t_{i+1}-t)}}{\theta} - \frac{b}{\theta^2} e^{\theta(t_{i+1}-t)} - \frac{(a+bt_i)}{\theta} + \frac{b}{\theta^2} \quad \text{Eqn (6)}$$

Ordering quantity for the i^{th} cycle

$$Q_{i+1}(t) = I_{i+1}(t_i)$$

$$Q_{i+1}(t) = \frac{(a+bt_{i+1})e^{\theta(t_{i+1}-t_i)}}{\theta} - \frac{b}{\theta^2} e^{\theta(t_{i+1}-t_i)} - \frac{(a+bt_i)}{\theta} + \frac{b}{\theta^2} \quad \text{Eqn (7)}$$

Retailer's overall cost

Total cost = Cost of ordering + Cost of inventory + purchasing cost + Deterioration cost

$$TCR_{NC} = n_1 * OC + \sum_{i=0}^{n_1-1} HC \int_{t_i}^{t_{i+1}} I_{i+1}(t) dt + \sum_{i=0}^{n_1-1} WP * Q_{i+1}(t) + \sum_{i=0}^{n_1-1} (DC * \theta) \int_{t_i}^{t_{i+1}} I_{i+1}(t) dt \quad \text{Eqn (8)}$$

$$TCR_{NC} = n_1 * OC + \sum_{i=0}^{n_1-1} HC + DC * \theta \int_{t_i}^{t_{i+1}} I_{i+1}(t) dt + \sum_{i=0}^{n_1-1} WP * Q_{i+1}(t) \quad \text{Eqn (9)}$$

$$TCR_{NC} = n_1 * OC + \sum_{i=0}^{n_1-1} HC + DC * \theta \int_{t_i}^{t_{i+1}} \left\{ \frac{(a+bt_{i+1})e^{\theta(t_{i+1}-t)}}{\theta} - \frac{b}{\theta^2} e^{\theta(t_{i+1}-t)} - \frac{(a+bt_i)}{\theta} + \frac{b}{\theta^2} \right\} dt + \sum_{i=0}^{n_1-1} WP * Q_{i+1}(t) \quad \text{Eqn (10)}$$

$$TCR_{NC} = n_1 * OC + \sum_{i=0}^{n_1-1} HC + DC * \theta \left\{ \frac{-(a+bt_{i+1})}{\theta^2} + \frac{b}{\theta^3} - \frac{at_{i+1}}{\theta} - \frac{b(t_{i+1})^2}{2\theta} + \frac{bt_{i+1}}{\theta^2} + \frac{(a+bt_{i+1})e^{\theta(t_{i+1}-t_i)}}{\theta^2} - \frac{b}{\theta^3} e^{\theta(t_{i+1}-t_i)} + \frac{at_i}{\theta} + \frac{bt_i^2}{2\theta} - \frac{bt_i}{\theta^2} \right\} + \sum_{i=0}^{n_1-1} WP * Q_{i+1}(t)$$

$$TCR_{NC} = n_1 * OC + \sum_{i=0}^{n_1-1} HC + DC * \theta \left\{ \frac{(a+bt_{i+1})e^{\theta(t_{i+1}-t_i)}}{\theta^2} - \frac{b}{\theta^3} e^{\theta(t_{i+1}-t_i)} - \frac{(a+bt_i)}{\theta^2} + \frac{b}{\theta^3} \right\} + \sum_{i=0}^{n_1-1} HC + DC * \theta \left\{ \frac{-at_{i+1}}{\theta} + \frac{at_i}{\theta} - \frac{b(t_{i+1})^2}{2\theta} + \frac{bt_i^2}{2\theta} \right\} + \sum_{i=0}^{n_1-1} WP \left\{ \frac{(a+bt_{i+1})e^{\theta(t_{i+1}-t_i)}}{\theta} - \frac{b}{\theta^2} e^{\theta(t_{i+1}-t_i)} - \frac{(a+bt_i)}{\theta} + \frac{b}{\theta^2} \right\}$$

$$TCR_{NC} = n_1 * OC + \sum_{i=0}^{n_1-1} \left\{ \frac{HC}{\theta} + DC + WP \right\} \left\{ \frac{(a+bt_{i+1})e^{\theta(t_{i+1}-t_i)}}{\theta} - \frac{b}{\theta^2} e^{\theta(t_{i+1}-t_i)} - \frac{(a+bt_i)}{\theta} + \frac{b}{\theta^2} \right\} - \sum_{i=0}^{n_1-1} HC + DC * \theta \left\{ \frac{a(t_{i+1}-t_i)}{\theta} + \frac{b(t_{i+1}^2-t_i^2)}{2\theta} \right\}$$

$$TCR_{NC} = n_1 * OC + \sum_{i=1}^{n_1-1} \left\{ \frac{HC}{\theta} + DC + WP \right\} \int_{t_i}^{t_{i+1}} (a+bt)e^{\theta(t-t_i)} dt - \left\{ \frac{HC}{\theta} + DC \right\} \{aH + 0.5bH^2\} \quad \text{Eqn (11)}$$

Table 1 presents the total cost incurred by the retailer when operating independently, as derived from the corresponding equations. Differentiation of eqn (11) w.r.t. t_i , we have,

$$\frac{\partial(TCR_{NC})}{\partial t_i} = \left\{ \frac{HC}{\theta} + DC + WP \right\} \left\{ (a+bt_i)(e^{\theta(t_i-t_{i-1})} - 1) - \theta \left\{ \frac{(a+bt_{i+1})e^{\theta(t_{i+1}-t_i)}}{\theta} - \frac{b}{\theta^2} e^{\theta(t_{i+1}-t_i)} - \frac{(a+bt_i)}{\theta} + \frac{b}{\theta^2} \right\} \right\} \quad \text{Eqn (12)}$$

$$\frac{\partial(TCR_{NC})}{\partial t_i} = \left\{ \frac{HC}{\theta} + DC + WP \right\} \left\{ (a+bt_i)e^{\theta(t_i-t_{i-1})} - (a+bt_{i+1})e^{\theta(t_{i+1}-t_i)} + \frac{b}{\theta} e^{\theta(t_{i+1}-t_i)} - \frac{b}{\theta} \right\} \quad \text{Eqn (13)}$$

Min $(TCR_{NC}) \{n_1, t_0, t_1, \dots, t_{n_1-1}\}$ at the optimal solution. Table 2 demonstrates the optimal solution $\{n_1^*, t_0^*, t_1^*, \dots, t_{n_1-1}^*\}$.

Again, differentiate the second time eqn (13) w.r.t. t_i , we have,

$$\frac{\partial^2(TCR_{NC})}{\partial t_i^2} = \left\{ \frac{HC}{\theta} + DC + WP \right\} \{e^{\theta(t_i-t_{i-1})}\{b + \theta(a+bt_i)\} + \theta(a+bt_{i+1})e^{\theta(t_{i+1}-t_i)} - be^{\theta(t_{i+1}-t_i)}\} \quad \text{Eqn (14)}$$

Second time differentiation of eqn (13) w.r.t. t_{i-1} , then we get,

$$\frac{\partial^2(TCR_{NC})}{\partial t_i \partial t_{i-1}} = \left\{ \frac{HC}{\theta} + DC + WP \right\} \{e^{\theta(t_i-t_{i-1})}(a+bt_i)(-\theta)\} \quad \text{Eqn (15)}$$

$$\frac{\partial^2(TCR_{NC})}{\partial t_i \partial t_{i-1}} = -\left\{ \frac{HC}{\theta} + DC + WP \right\} \theta e^{\theta(t_i-t_{i-1})}(a+bt_i) \quad \text{Eqn (16)}$$

Again, second time differentiate eqn (13) w.r.t. to t_{i+1} , then we have,

$$\frac{\partial^2(TCR_{NC})}{\partial t_i \partial t_{i+1}} = \left\{ \frac{HC}{\theta} + DC + WP \right\} \left\{ -\{be^{\theta(t_{i+1}-t_i)} + (a+bt_{i+1})e^{\theta(t_{i+1}-t_i)}\theta\} + \frac{b}{\theta} e^{\theta(t_{i+1}-t_i)} \right\}$$

$$\frac{\partial^2(TCR_{NC})}{\partial t_i \partial t_{i+1}} = -\left\{ \frac{HC}{\theta} + DC + WP \right\} \theta e^{\theta(t_{i+1}-t_i)}(a+bt_{i+1}) \quad \text{Eqn (17)}$$

$$\frac{\partial^2(TCR_{NC})}{\partial t_i \partial n} = 0; \forall n, \text{ different from } i, i+1, i-1. \quad \text{Eqn (18)}$$

If the Hessian matrix of TCR is positive definite, it confirms that the aggregate cost function is at its minimum.

$$\nabla^2 TCR = \begin{bmatrix} \frac{\partial^2 TCR}{\partial t_1^2} & \frac{\partial^2 TCR}{\partial t_1 \partial t_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\partial^2 TCR}{\partial t_2 \partial t_1} & \frac{\partial^2 TCR}{\partial t_2^2} & \frac{\partial^2 TCR}{\partial t_2 \partial t_3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial^2 TCR}{\partial t_3 \partial t_2} & \frac{\partial^2 TCR}{\partial t_3^2} & \frac{\partial^2 TCR}{\partial t_3 \partial t_4} & 0 & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial^2 TCR}{\partial t_{n_1-1} \partial t_{n_1-2}} & \frac{\partial^2 TCR}{\partial t_{n_1-1}^2} & \frac{\partial^2 TCR}{\partial t_{n_1-1} \partial t_{n_1}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial^2 TCR}{\partial t_{n_1} \partial t_{n_1-1}} & \frac{\partial^2 TCR}{\partial t_{n_1}^2} \end{bmatrix}$$

Fig 2: Hessian matrix Nitin Kumar Mishra *et al.* ^[16]

Condition to check positive definite

$$\left| \frac{\partial^2 TCR_{NC}(n_1, t_0, t_1, \dots, t_n)}{\partial t_i^2} \right| \geq \left| \frac{\partial^2 TCR_{NC}(n_1, t_0, t_1, \dots, t_n)}{\partial(t_i)(t_{i-1})} \right| + \left| \frac{\partial^2 TCR_{NC}(n_1, t_0, t_1, \dots, t_n)}{\partial(t_i)(t_{i+1})} \right| \quad \text{Eqn (19)}$$

$$\left| \frac{\partial^2 TCR_{NC}(n_1, t_0, t_1, \dots, t_n)}{\partial t_i^2} \right| - \left| \frac{\partial^2 TCR_{NC}(n_1, t_0, t_1, \dots, t_n)}{\partial(t_i)(t_{i-1})} \right| - \left| \frac{\partial^2 TCR_{NC}(n_1, t_0, t_1, \dots, t_n)}{\partial(t_i)(t_{i+1})} \right| \geq 0 \quad \text{Eqn (20)}$$

$$\left\{\frac{HC}{\theta} + DC + WP\right\}\left\{e^{\theta(t_i-t_{i-1})}\{b + \theta(a + bt_i)\} + \theta(a + bt_{i+1})e^{\theta(t_{i+1}-t_i)} - be^{\theta(t_{i+1}-t_i)}\right\} + \left\{\frac{HC}{\theta} + DC + WP\right\}\theta e^{\theta(t_i-t_{i-1})}(a + bt_i) + \left\{\frac{HC}{\theta} + DC + WP\right\}\theta e^{\theta(t_{i+1}-t_i)}(a + bt_{i+1}) \geq 0 \quad \text{Eqn (21)}$$

Theorem: - If t_i satisfy inequations (i) $\left|\frac{\partial^2 TCR_{NC}(n_1, t_0, t_1, \dots, t_{n_1})}{\partial t_i^2}\right| \geq 0$

$$(ii) \left|\frac{\partial^2 TCR_{NC}(n_1, t_0, t_1, \dots, t_n)}{\partial t_i^2}\right| \geq \left|\frac{\partial^2 TCR_{NC}(n_1, t_0, t_1, \dots, t_n)}{\partial (t_i)(t_{i-1})}\right| + \left|\frac{\partial^2 TCR_{NC}(n_1, t_0, t_1, \dots, t_n)}{\partial (t_i)(t_{i+1})}\right|$$

for all $i = 1, 2, 3, \dots, n_1-1$. Then $\nabla^2 TCR_{NC}$ is positive definite.

When coordination is not present, the supplier's overall expenses depend entirely on the replenishment decisions made by the retailer. Therefore, throughout the planning horizon T , the supplier incurs only two major types of costs: the cost of setting up production and the cost of manufacturing the ordered quantity.

Supplier total cost:

$$TCS_{NC} = n_1 * SC + \sum_{i=0}^{n_1-1} PC * Q_{i+1}^* \quad \text{Eqn (22)}$$

Additionally, over the entire planning horizon T , the optimum ordered quantity is determined by

$$Q_{i+1} = \sum_{i=0}^{n_1-1} PC * Q_{i+1}^* \quad \text{Eqn (23)}$$

Refer to Tables 3 and 4 to view the total cost for suppliers and the optimal replenishment quantity without coordination.

Model of Centralized Strategic Planning Under a Specified Period

In a centralised decision, the process of cooperation is initiated by the supplier. With this suitable coordination in the inventory supply chain, we were able to achieve our objective, i.e. to minimize overall cost. All the participants of the supply chain collaborate to make a single decision to optimize the inventory cost in the supply chain process. This contribution of all the partners fully depends on the types of goods, the demand for goods, deterioration and other factors. Consequently, the overall supply chain is affected in a centralized system.

$$TCS_{WC} = n_2 * SC + n_2 * OC + \sum_{j=0}^{n_2-1} \{DC * \theta + HC\} \int_{t_j}^{t_{j+1}} I'_{j+1}(t) dt + \sum_{j=0}^{n_2-1} \{WP + PC\} Q'_{j+1} - \text{Min } TCR_{NC} \quad \text{Eqn (24)}$$

$$TCS_{WC} = n_2 * \{SC + OC\} + \sum_{j=0}^{n_2-1} \{DC * \theta + HC\} \int_{t_j}^{t_{j+1}} I'_{j+1}(t) dt + \sum_{j=0}^{n_2-1} WP * Q'_{j+1} + \sum_{j=0}^{n_2-1} PC * Q'_{j+1} - \text{Min } TCR_{NC}$$

Where $t'_0 = 0$ and $t'_{n_2-1} = T$

Eqn (25)

$\text{Min } (TCS_{WC}) \{n_1, t_0, t_1, \dots, t_{n_1-1}\}$ at the optimal solution $\{n_1^*, t_0^*, t_1^*, \dots, t_{n_1-1}^*\}$.

$\text{Min } (TCS_{WC}) \{n_2, t'_0, t'_1, \dots, t'_{n_2-1}\}$ at the optimal solution $\{n_2, t'^*_0, t'^*_1, \dots, t'^*_{n_2-1}\}$

Additionally, over the entire planning horizon T , the order quantity is determined by

$$Q_{j+1} = \sum_{j=0}^{n_2-1} PC * Q_{j+1}^* \quad \text{Eqn (26)}$$

$$TCS_{WC} = n_2 * OC + \sum_{j=0}^{n_2-1} WP * Q_{j+1} + \sum_{j=0}^{n_2-1} \{DC * \theta + HC\} \int_{t_j}^{t_{j+1}} I_{i+1}(t) dt$$

$$TCS_{WC} = n_2 * OC + \sum_{j=0}^{n_2-1} \{DC * \theta + HC\} \int_{t_j}^{t_{j+1}} \frac{D(t)e^{\theta(t_{i+1}-t)} - 1}{\theta} dt + \sum_{j=0}^{n_2-1} WP * \frac{D(t)e^{\theta(t_{i+1}-t_i)} - 1}{\theta}$$

The cost of generating the systems has been reduced. The retailer and the supplier divide the profit earned from shorter replenishment cycles. Profit earned:

$$P = \{\{TCR_{NC} + TCS_{NC}\} - \{TCR_{WC} - TCS_{WC}\}\}$$

Retailer's Achieved Cost:

$$TCR_{WC}^P = \{TCR_{NC}(n_1) - \frac{TCR_{NC}(n_1)}{TCR_{NC}(n_1) + TCS_{NC}(n_1)}\} * P$$

$$\text{Supplier's Achieved Cost: } TCS_{WC}^P = \{TCS_{NC}(n_1) - \frac{TCS_{NC}(n_1)}{TCR_{NC}(n_1) + TCS_{NC}(n_1)}\} * P$$

$$\text{Now percentage gain in the Retailer's cost} = \frac{\text{TCR}_{\text{WC}}^{\text{P}}}{\text{TCR}_{\text{NC}}(n_1)} * 100$$

$$\text{Also, the percentage gain in the Supplier's cost} = \frac{\text{TCS}_{\text{WC}}^{\text{P}}}{\text{TCS}_{\text{NC}}(n_1)} * 100$$

Algorithm

1. First, assign constant values to all the given parameters: PC, WP, θ , x, y, z.
2. To identify the most efficient ordering strategy under a decentralized decision-making environment.
 - a. If we will take, $n_1 = 1$ then $t_0 = 0$ and $t_1 = T$.
 - b. If we take $n_1 = 2$, after that, assign the initial value of $t_0 = 0$ and $t_2 = T$. Then, we can calculate t_1 by using eqn (13).
 - c. Using eqn (13), compute t_2 based on the value of t_1 obtained in the previous step.
 - d. Next, use the value of t_2 to calculate t_3 from eqn (13). Continue this process in the same manner until t_{n_1-1} is determined.
 - e. When the value of t_{n_1-1} is approximately equal to the planning horizon T, and the computed values of t_i satisfy the condition of the Hessian matrix, then the process is complete.
 - f. For each $n_1 = 1, 2, 3, \dots$, we will calculate the unique and optimal values of t_i .
 - g. Using eqn (11), compute $\text{TCR}_{\text{NC}}(n_1)$ to determine the retailer's optimal total cost. This is evaluated based on the following condition:
 - h. For $n_1 = 1$, $\text{TCR}_{\text{NC}} = \text{TCR}_{\text{NC}}(n_1)$ and stop for $n_1 \geq 2$, and if $\text{TCR}_{\text{NC}}(n_1) \leq \text{TCR}_{\text{WC}}(n_1-1)$ and $\text{TCR}(n_1) \leq \text{TCR}(n_1+1)$ then $\text{TCR}_{\text{NC}}(n_1) = \text{Optimal (TCR)}$ and then stop; otherwise, go to the previous step.
 - Similarly, we can obtain the value of $\text{TCS}_{\text{NC}}(n_1)$ and $Q_{i+1}(n_1)$ system by using eqns (22) and (23).

Tabular Representation and Formulation

Table 1: Total Cost of the Retailer without Coordination for Five Different Values of ‘a’

$\rightarrow n_1$ $\downarrow a$	1	2	3	4	5	6	7
0.40	60.9233	108.729	101.614	81.8785	67.1308	59.3736	56.837
0.32	57.9125	104.726	98.511	79.8469	65.8544	58.5975	56.3565
0.36	59.4192	106.727	100.06	80.8638	66.489	58.9866	56.5957
0.44	62.425	110.734	103.172	82.9066	67.7664	59.7662	57.0803
0.48	63.9245	112.741	104.735	83.938	68.4089	60.161	57.3253

Table 2: The lowest total cost of the retailer and the optimal number of replenishment cycles without coordination

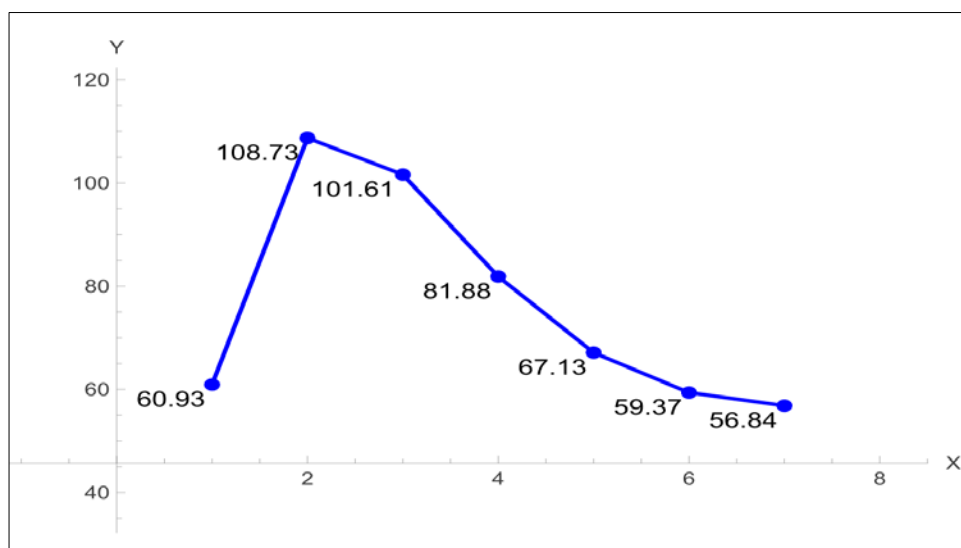
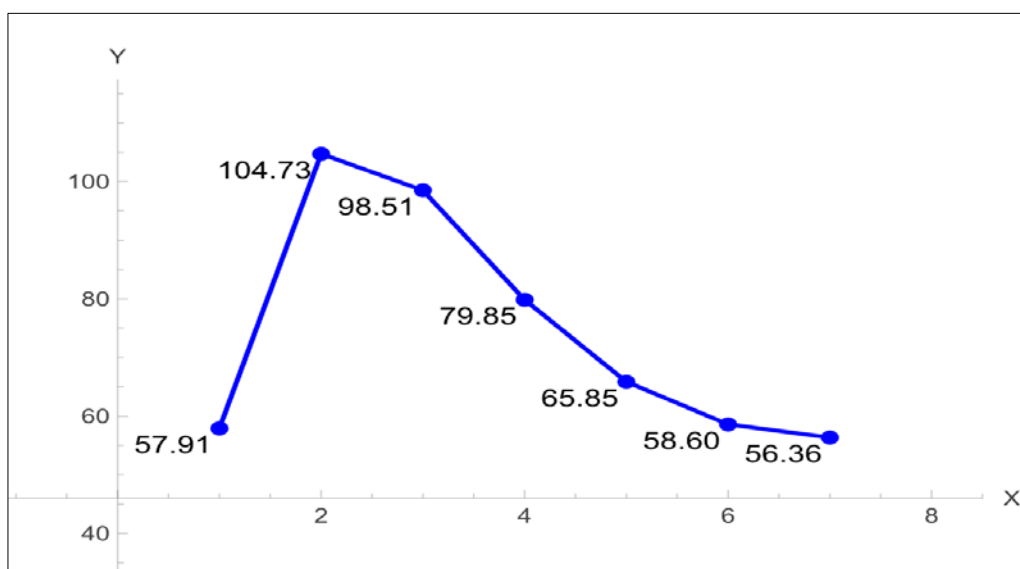
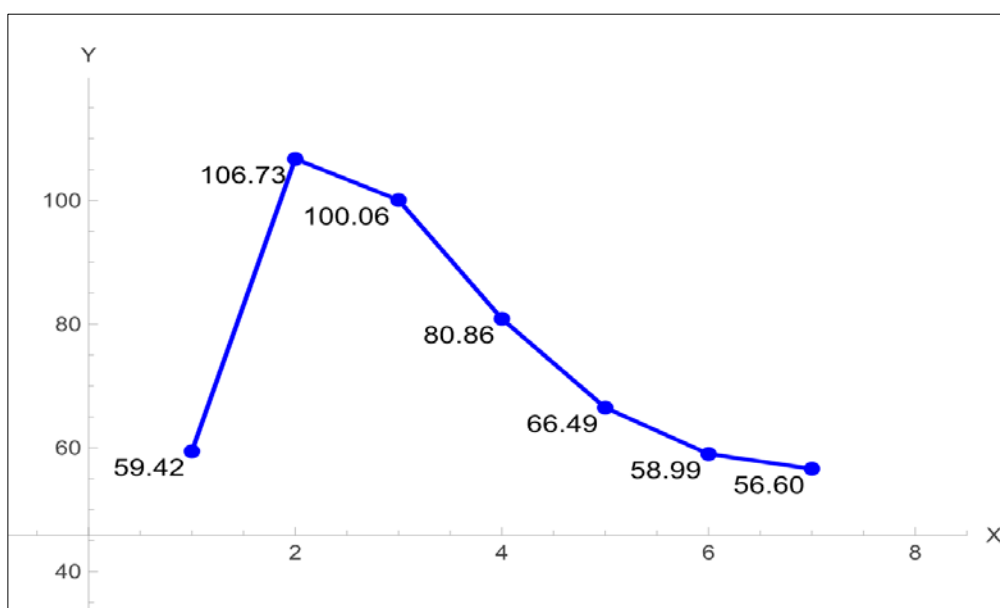
a	n	t_0^*	t_1^*	t_2^*	t_3^*	t_4^*	t_5^*	t_6^*	TCR_{NC}
0.40	7	0	0.651165	1.22678	1.75437	2.24752	2.7142	4	56.837
0.32	7	0	0.663387	1.24254	1.77017	2.26158	2.7255	4	56.3565
0.36	7	0	0.657042	1.2344	1.76203	2.25435	2.7197	4	56.5957
0.44	7	0	0.645709	1.21964	1.74715	2.24105	2.70898	4	57.0803
0.48	7	0	0.640631	1.21293	1.74032	2.23491	2.704	4	57.3253

Table 3: Total cost for the supplier without coordination for five different values of ‘a’

a	TCS_{NC}
0.40	84.0191
0.32	84.0195
0.36	84.0193
0.44	84.0190
0.48	84.0188

Table 4: Optimal replenishment quantity without coordination

a	Quantity
0.40	3.07128
0.32	2.95921
0.36	3.01503
0.44	3.12792
0.48	3.18491

Total cost of the retailer for different values of 'a'**Fig 3:** Total retailer cost graph for $a = 0.40$ **Fig 4:** Total retailer cost graph for $a = 0.32$ **Fig 5:** Total retailer cost graph for $a = 0.36$

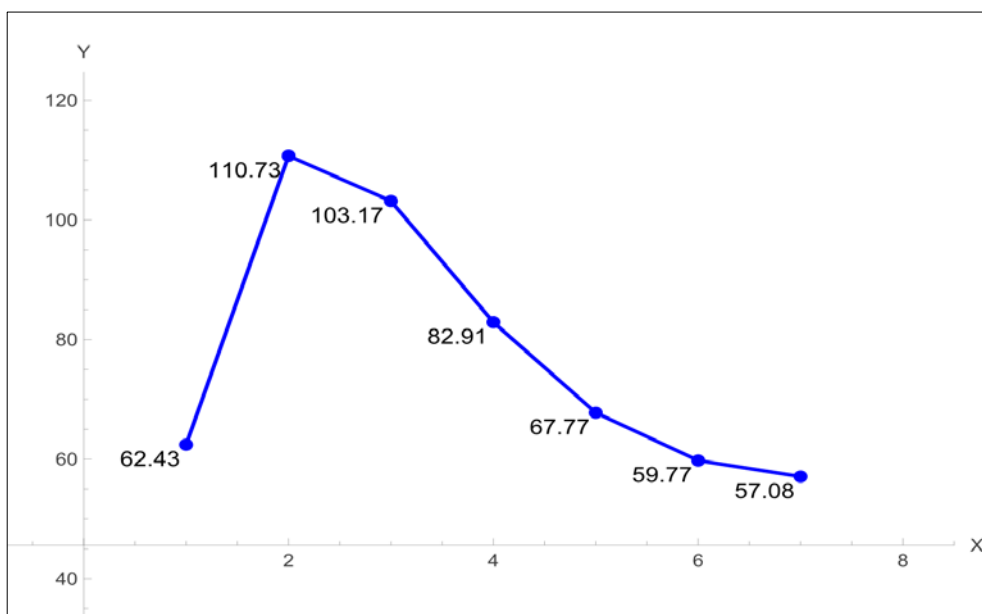


Fig 6: Total retailer cost graph for $a = 0.44$

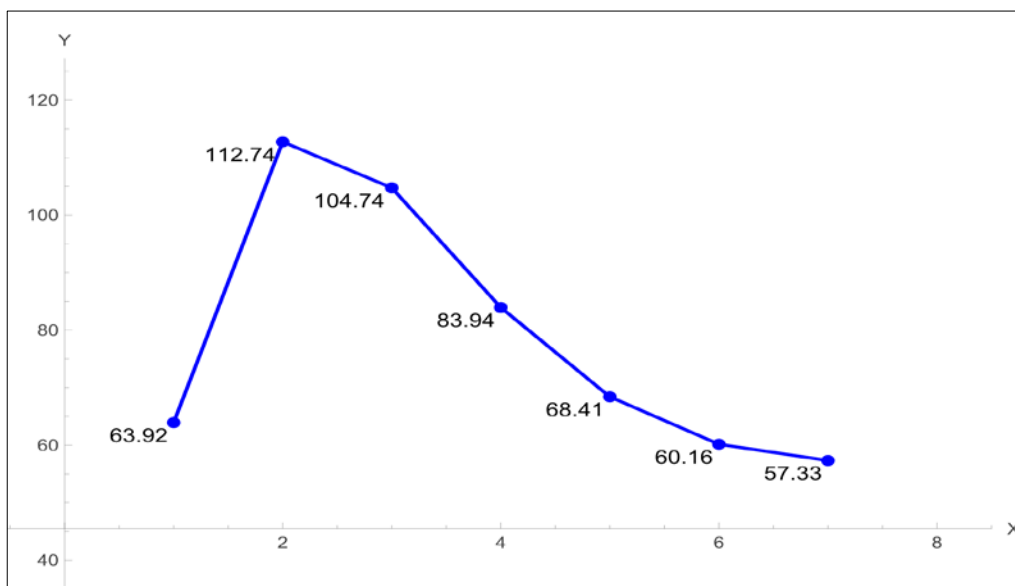


Fig 7: Total retailer cost graph for $a = 0.48$

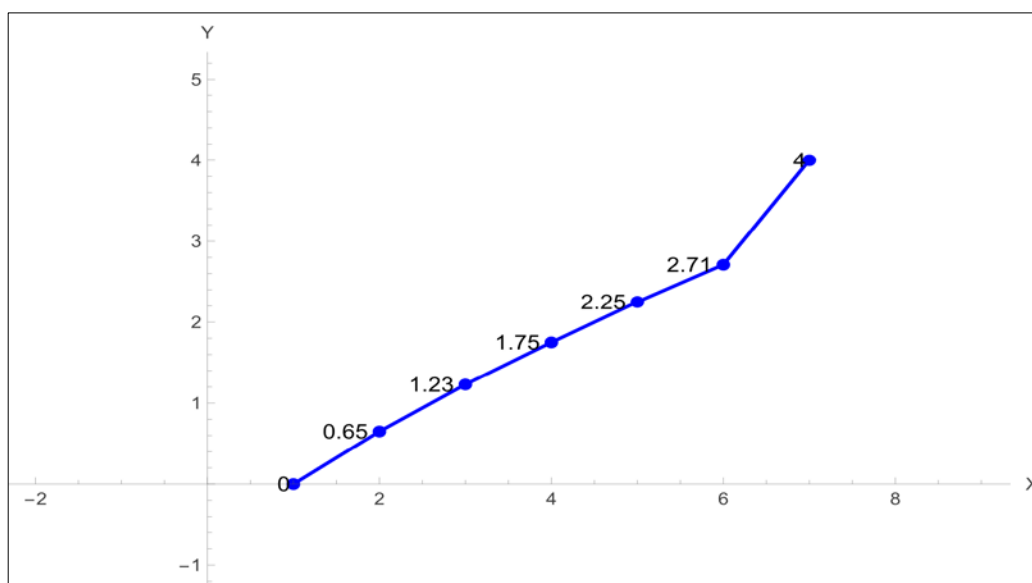


Fig 8: The lowest total cost of the retailer graph for $a = 0.40$

Conclusion

The paper presents a coordinated inventory model that incorporates deterioration in the supply chain framework. This model helps make perfect decisions across many aspects, such as inventory management, product selection, and, most importantly, minimizing product costs. Effective coordination between the supplier and retailer helps to reduce the overall system cost. In the present paper, by using computational tools such as Mathematica, we calculate the total cost of the supplier, retailer and the optimal replenishment quantity. A comparative evaluation of essential variables accommodates the managerial perceptiveness into the impact of deterioration rates on system performance. Graphical illustrations are also included to enhance the clarity and understanding of the model. While controlling the rate of deterioration, it helps to minimize the cost of production and also helps to improve the effectiveness of the inventory supply process.

Additionally, our research can be extended by including such factors as trade credit policies, demand dependent on pricing and advertising, and fuzzy factors. Although we can extend this model by using a three-echelon supply chain and multiple retailers, also. Overall, this study contributes a beneficial framework for effective stock management and decision-making in the supply chain.

References

1. Lee HL, Billington C. Material Management in Decentralized Supply Chains. *Operations Research*. 1993;41:835-847.
2. Whitin TM. The theory of inventory management. 2nd ed. Princeton: Princeton University Press; 1957.
3. Dye CY, Ouyang LY, Hsieh TP. Deterministic inventory model for deteriorating items with capacity constraint and time-proportional backlogging rate. *European Journal of Operational Research*. 2007;178(3):789-807.
4. Mishra VK, Singh LS. Deteriorating inventory model for time-dependent demand and holding cost with partial backlogging. *International Journal of Management Science and Engineering Management*. 2011;6(4):267-271.
5. Mahapatra AS, Mahapatra MS, Sarkar B, Majumder SK. Benefit of preservation technology with promotion and time-dependent deterioration under fuzzy learning. *Expert Systems Applications*. 2022;201:117169.
6. Chang CT, Teng JT, Goyal SK. Optimal replenishment policies for non-instantaneous deteriorating items with stock-dependent demand. *International Journal of Production Economics*. 2010;123(1):62-68.
7. Goyal SK, Giri BC. Recent trends in modelling of deteriorating inventory. *European Journal of Operational Research*. 2001;134(1):1-16.
8. Yang HL. A comparison among various partial backlogging inventory lot-size models for deteriorating items based on maximum profit. *International Journal of Production Economics*. 2005;96(1):119-128.
9. Pal B, Sana SS, Chaudhuri K. A multi-echelon supply chain model for reworkable items in multiple markets with supply disruption. *Economic Modelling*. 2012;29:1891-1898.
10. Widvadana G, Wee HM. An economic production quantity model for deteriorating items with preventive maintenance policy and random machine breakdown. *International Journal of System Science*. 2012;43:1870-1882.
11. Skouri K, Konstantaras I, Papachristos S, Ganas I. Inventory models with ramp-type demand rate, partial backlogging and Weibull deterioration rate. *European Journal of Operational Research*. 2009;192(1):79-92.
12. Chang HJ, Hung CH, Dye CY. An inventory model for deteriorating items with linear trend demand under the condition of permissible delay in payments. *Production Planning & Control*. 2001;12(3):274-282.
13. Jaggi CK, Gupta M, Kausar A, Tiwari S. Inventory and credit decisions for deteriorating items with displayed stock dependent demand in a two-echelon supply chain using Stackelberg and Nash equilibrium solution. *Annals of Operations Research*. 2019;274:309-329.
14. Hung KC. An inventory model with generalised type demand, deterioration and backorder rates. *European Journal of Operational Research*. 2011;208(3):239-242.
15. Mishra NK, Jain P. Blockchain-enhanced inventory management in decentralized supply chains for finite planning horizons. [Journal details missing].
16. Mishra NK, Jain P. Blockchain-Enhanced Inventory Management in Decentralized Supply Chains for Finite Planning Horizons. *Journal Européen des Systèmes Automatisés*. 2024;57(1).