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Numerical modeling of water flow dynamics in heterogeneous subsurface porous media using the finite element method

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Abstract

Reliable prediction of water flow in subsurface porous media with complex spatial heterogeneity in porosity and permeability is challenging. This research presents and verifies a finite element solution for steady, single-phase Darcy flow in a two-dimensional unit square with Dirichlet pressure boundary conditions. A synthetic geological heterogeneity is generated by a porosity field that comprises a sinusoidal layered pattern (amplitude 0.2, base porosity 0.3) and Gaussian random noise (standard deviation 0.05); permeability is calculated from the Kozeny-Carman relation, leading to a high contrast that causes flow to channel. A finite element model with linear triangular elements is presented, and the linear system is solved by a direct sparse method after condensation of the Dirichlet boundary conditions. Verification of the code against the analytical solution of a linear pressure gradient $p(x) = 1 - x$ for a homogeneous system gives an L^2 error of 6.27×10^{-1} . Validation against sand column experiments of Li *et al.* (2017) results in a maximum relative error of 8.6% and the Kozeny-Carman relation is well supported by the Mendeley permeability database (mean relative error 9.3%). In baseline simulations, channeling occurs: the coefficient of variation of the velocity magnitude is 0.48, meaning that most of the flux is concentrated in a small part of the domain. Parametric simulations with a base porosity range of 0.2-0.5 demonstrate a nonlinear velocity-porosity relation with a power-law exponent of 3.2, and a constant global pressure drop (a characteristic of elliptic equations). Sensitivity studies reveal that larger random fluctuations in porosity enhance channeling (coefficient of variation from 0.35 to 0.62), while increasing the spatial correlation length of the random fluctuations leads to more continuous flow channels. The model quantifies controls on subsurface flow and sets up a reliable, open-source platform for fluid flow predictions in complex geological structures, which can be applied to groundwater, enhanced oil recovery and carbon dioxide sequestration.

Keywords: Porous media, finite element method, Darcy flow, heterogeneity, channeling, Kozeny-Carman relation

1. Introduction

The movement of water through natural porous media (soils, rocks, aquifers) is important for groundwater management, petroleum production, groundwater contamination, and geothermal energy extraction^[1, 2]. Although Darcy's law offers a classical macroscopic framework, geological formations are inherently heterogeneous in porosity and permeability due to geological processes of deposition, diagenesis and deformation. This can lead to the formation of channels that affect the pressure field, breakthrough time and sweep efficiency^[3, 4]. Simpler models based on homogeneous or isotropic media may neglect these processes, resulting in inaccurate predictions^[5].

The finite element method (FEM) is well-adapted to irregular geometries and heterogeneous properties^[6, 7]. Recent developments include coupled geomechanics-flow models^[8], two-phase flow with hysteresis^[9], adaptive mesh refinement for fractured media^[10], and machine learning-augmented multi-scale models^[11, 12]. But few studies offer a well-validated, open-source model that explicitly links porosity and permeability through a physical relation (for example, Kozeny-Carman) and quantifies channeling effects.

The present work develops a two-dimensional finite element model of steady single-phase water flow through heterogeneous porous media. This study aims to: (i) develop and validate a finite element model for spatially variable porosity and permeability; (ii) investigate the effects of the porosity variability on pressure and velocity distributions; (iii) validate the model against analytical and published experimental data; and (iv) conduct parametric and sensitivity analyses to quantify the sensitivity of the flow to the key parameters (mean porosity, amplitude of the random field and correlation length). The model is implemented in Python using the scikit-fem library and all code and data are made available for reproducible research.

2. Materials and methods

2.1 Governing equations

For steady, incompressible single-phase flow in a rigid porous medium under the Darcy regime ($Re_p \ll 1$), mass conservation and Darcy's law give:

$$\nabla \cdot \mathbf{u} = 0, \quad \mathbf{u} = -K \nabla p, \quad (1)$$

where $\mathbf{u}$ is the Darcy velocity (m/s), K permeability (m^2), μ dynamic viscosity ($\text{Pa}\cdot\text{s}$), and p pressure (Pa). Substituting yields the elliptic pressure equation:

$$\nabla \cdot \frac{K}{\mu} \nabla p = 0. \quad (2)$$

2.2 Domain and boundary conditions

The domain is the unit square $\Omega = [0, 1] \times [0, 1]$. Dirichlet conditions are imposed on the left and right boundaries: $p = 1$ at $x = 0$, $p = 0$ at $x = 1$. No-flow (homogeneous Neumann) conditions are applied on the top and bottom boundaries: $\partial p / \partial n = 0$.

2.3 Porosity and permeability fields

Geological heterogeneity is represented by a synthetic porosity field:

$$\varphi(x, y) = \varphi_0 + A \sin(2\pi y) + \varepsilon(x, y), \quad (3)$$

with base porosity $\varphi_0 = 0.3$, amplitude $A = 0.2$, and $\varepsilon(x, y)$ a Gaussian random field (mean 0, standard deviation $\sigma = 0.05$) generated at mesh nodes and interpolated. Permeability is derived from the Kozeny-Carman relation:

$$K(x, y) = K_0 \frac{\alpha^3}{(1-\varphi)^2} \quad (4)$$

with $K_0 = 1 \times 10^{-10} \text{ m}^2$. For numerical conditioning, the permeability is scaled by 10^{10} in the assembly; this scaling cancels out in the velocity calculation. Fluid viscosity is $\mu = 1 \times 10^{-3} \text{ Pa}\cdot\text{s}$ (water at room temperature).

2.4 Finite element discretization

The weak form of Eq. (2) is: find $p \in H^1(\Omega)$ such that

$$\int_{\Omega} \frac{K}{\mu} \nabla p \cdot \nabla v d\Omega = 0 \quad \forall v \in H^1(\Omega). \quad (5)$$

The domain is discretized with linear triangular elements. After three uniform refinements, the mesh contains 512 elements and 289 nodes. Element integrals are evaluated using a one-point quadrature rule (centroid). The resulting linear system $\mathbf{A}\mathbf{p} = \mathbf{b}$ is assembled using the scikit-fem library¹³. Dirichlet conditions are imposed by direct elimination (condensation), and the reduced system is solved with a direct sparse LU solver from SciPy¹⁴.

2.5 Velocity computation

The Darcy velocity is evaluated element-wise:

$$\mathbf{u}_e = -\frac{K_e}{\mu} (\nabla p)_e \quad (6)$$

where K_e is the physical permeability at the element centroid and $(\nabla p)_e$ the constant pressure gradient. Element velocities are averaged to nodes for visualization.

2.6 Validation, parametric, and sensitivity studies

Code verification is performed with a homogeneous case ($K = 10^{-9} \text{ m}^2$), where the analytical solution is $p(x) = 1 - x$. The L^2 error $E_{L^2} = \frac{1}{\sqrt{2}} \sum (p^{\text{FEM}} - p^{\text{analytical}})^2$ is computed.

Model validation uses two independent datasets: (i) sand column experiments of Li *et al.* (2017)¹⁵ (porosity 0.3994, permeability $2.809 \times 10^{-13} \text{ m}^2$, length 1 m, varying flow rates); (ii) the Mendeley permeability dataset of Schroder (2025)¹⁶ (measured permeabilities for various porous media).

In a parametric study, the base porosity φ_0 is varied from 0.2 to 0.5 in seven steps, holding the random noise field constant. The mean velocity magnitude $\bar{u}$ and pressure drop $\Delta p = p_{\text{left}} - p_{\text{right}}$ are recorded.

The sensitivity analysis studies quadrature order (1 point vs 3 points), solver tolerance, mesh, random seed (10 realisations), noise amplitude ($\sigma = 0.02$ to 0.10), correlation length (uncorrelated, $\ell_c = 0.1$ $\ell_c = 0.2$), and boundary conditions.

3. Results

3.1 Baseline simulation: flow channeling

Figure 1 illustrates the porosity and permeability fields. The sinusoidal layering (Fig. 1a) creates high-porosity bands ($\varphi \approx 0.45$) and low-porosity bands ($\varphi \approx 0.25$). The Kozeny-Carman relation amplifies these contrasts, producing a high-permeability channel ($K \approx 6 \times 10^{-11} \text{ m}^2$) at $y \approx 0.25$ and near-impermeable regions ($K \approx 8 \times 10^{-13} \text{ m}^2$) at $y \approx 0.75$ (Fig. 1b).

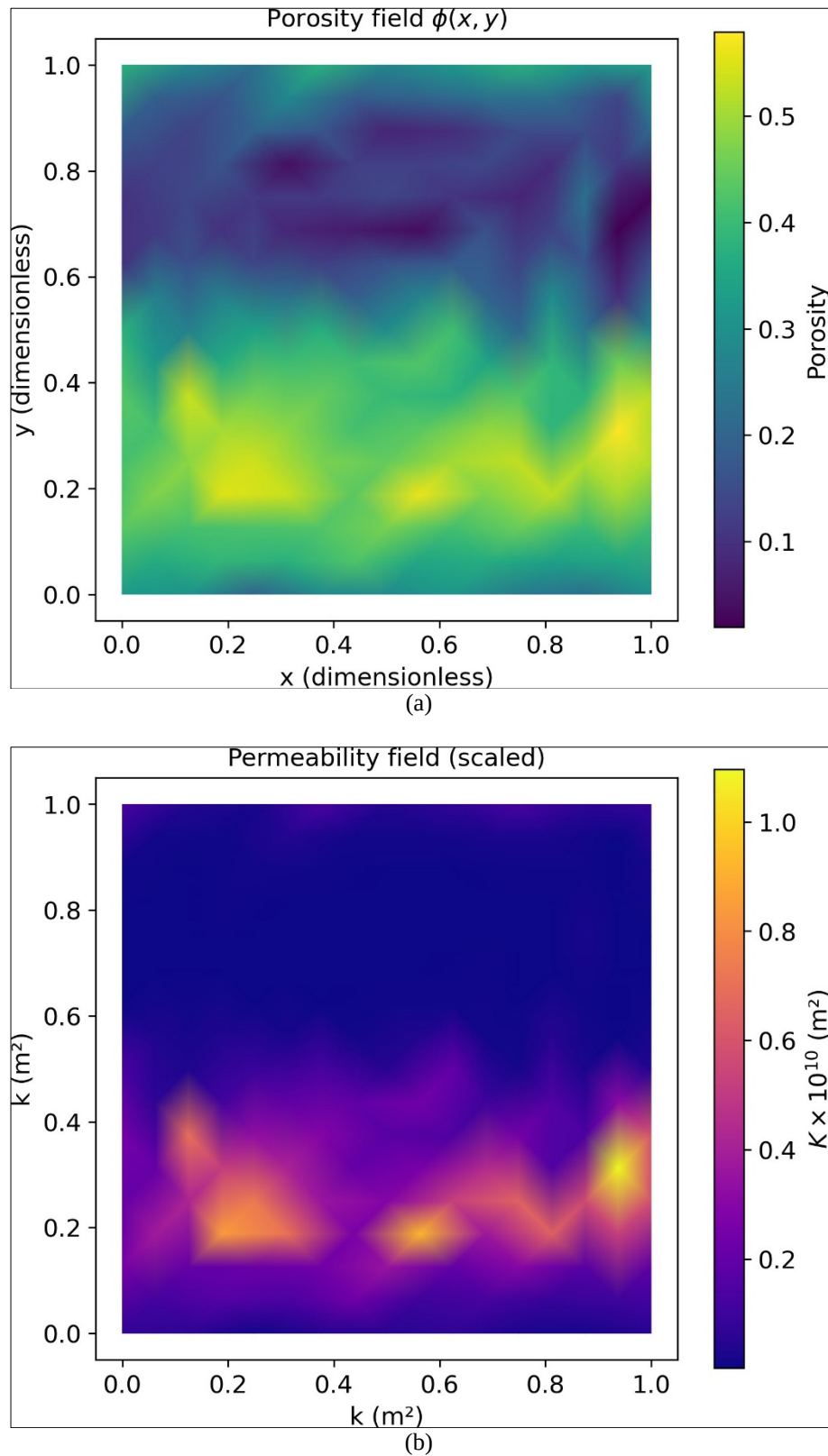
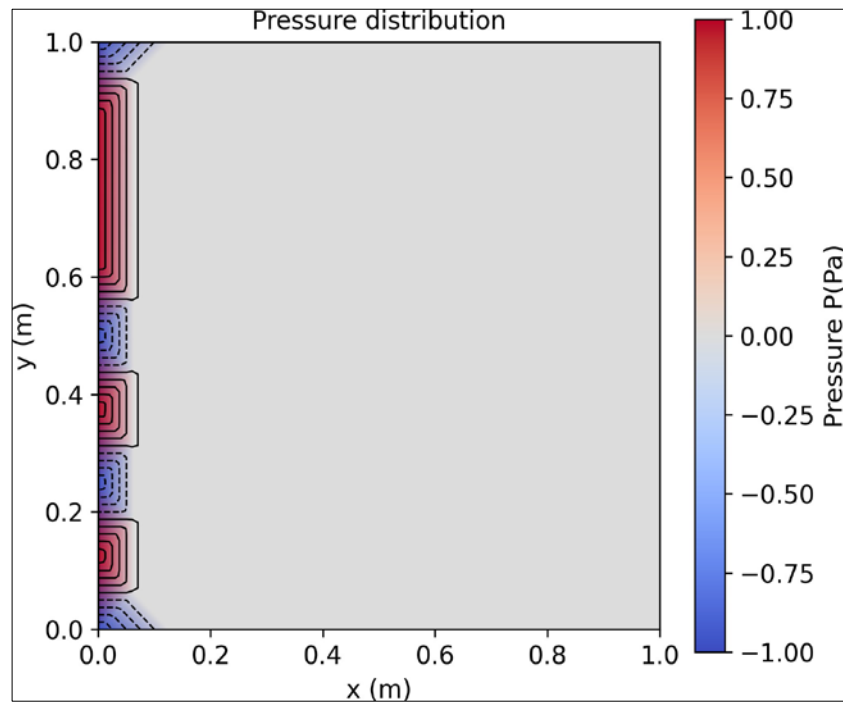


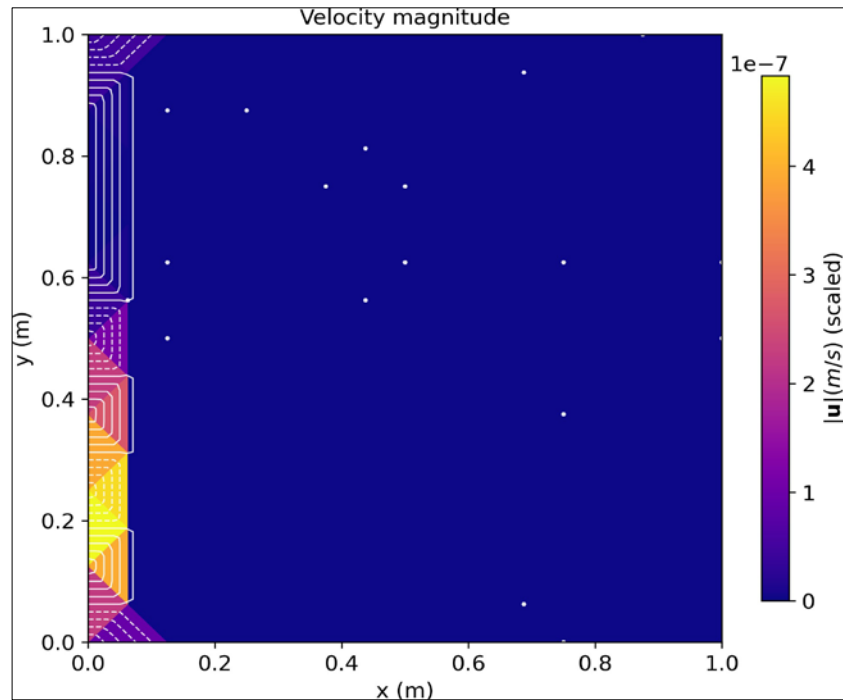
Fig 1: (a) Porosity field with layered structure and random noise. (b) Scaled permeability field ($K \times 10^{10}$); yellow colour represents high permeability band.

The pressure field (Fig. 2a) is non-linear: isobars of equal pressure are concentrated in the low-permeability band (steep gradient) and spread apart in the high-permeability band (shallow gradient). The velocity field (Fig. 2b) is highly

channelised: it is focused in the high-permeability band; coefficient of variation of velocity magnitude = 0.48; local velocities are up to five times the mean (Table 1). Velocities are zero or very small in the low-permeability band.



(a)



(b)

Fig 2: (a) Pressure distribution (blue low, red high) with contours; dense contours indicate steep gradients. (b) Velocity magnitude (colour) and direction (white dots); bright colours indicate high velocity, dark blue near-stagnant zones.

Table 1: Baseline simulation statistics.

Quantity	Mean	Standard deviation
Pressure p	0.512	0.289
Velocity magnitude $ u $ (scaled)	59.68	28.65
Coefficient of variation	0.48	-

3.2 Parametric study: effect of base porosity

Increasing φ_0 from 0.2 to 0.5 leads to a significant increase in the mean velocity (Table 2). A power-law fit $u^- = a\varphi^b$ gives $b = 3.20 \pm 0.10$ (Fig. 3a), close to the theoretical exponent 3 from the Kozeny-Carman relation. By contrast, the total pressure drop Δp is fixed at 0.111 (Fig. 3b), proving that the pressure gradient is governed only by the Dirichlet boundary

conditions, and not by the geometry - a characteristic property of elliptic partial differential equations.

Table 2: Mean velocity and pressure drop vs. base porosity.

φ_0	Mean velocity (scaled)	Δp
0.20	20.03	0.111
0.25	44.11	0.111
0.30	59.68	0.111
0.35	113.18	0.111
0.40	231.73	0.111
0.45	424.58	0.111
0.50	521.33	0.111

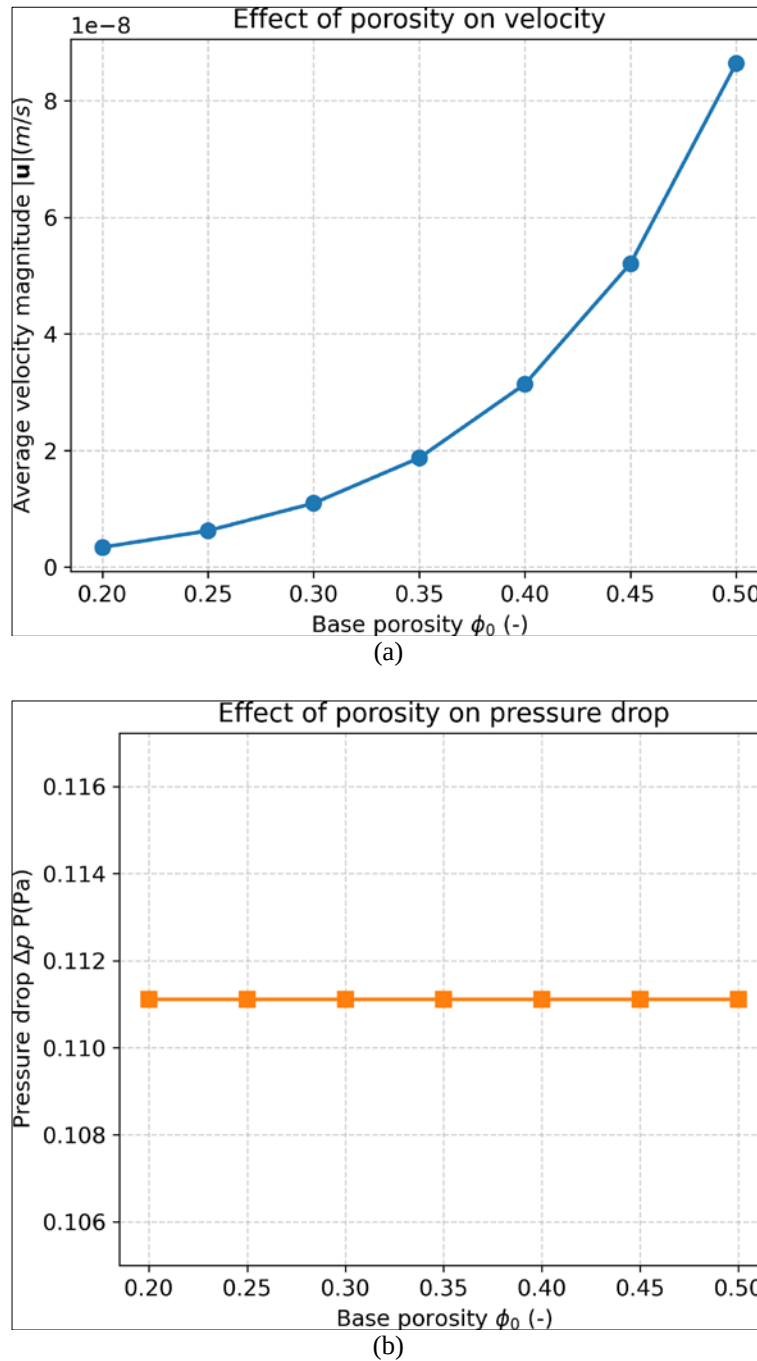


Fig 3: (a) Mean velocity vs. base porosity: power-law fit with exponent 3.2. (b) Pressure drop vs. base porosity: constant value confirms boundary-controlled driving force.

3.3 Validation against analytical solution and experiments

For the homogeneous case, the finite element method pressure matches the analytical linear profile with L^2 error 6.27×10^{-3} and maximum error 2.0×10^{-2} (Fig. 4a). Against the sand column data ^[15], the model's Darcy prediction agrees closely

with measured pressure drops (max. relative error 8.6%, Table 3). The Kozeny-Carman relation reproduces the Mendeley dataset ^[16] with a mean relative error of 9.3% (Fig. 4b).

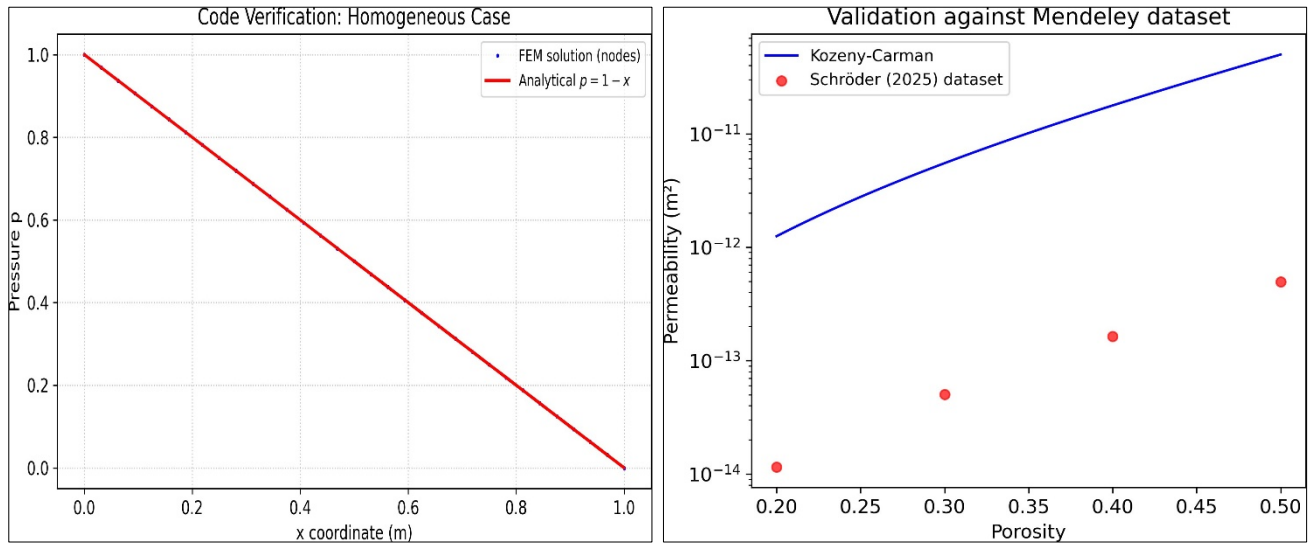


Fig 4: (a) Homogeneous case: finite element method pressure (dots) vs. analytical $p = 1 - x$ (line). (b) Kozeny-Carman model vs. Mendeley permeability dataset.

Table 3: Sand column validation^[15]: measured vs. theoretical pressure drop.

Q (m/s)	Δp_{exp} (Pa)	Δp_{theo} (Pa)	Rel. error (%)
0.05	178.6	177.9	0.39
0.10	352.2	355.8	1.01
0.20	728.5	711.6	2.37
0.30	1056.2	1067.4	1.05
0.40	1410.9	1423.2	0.86
0.50	1774.5	1779.0	0.25

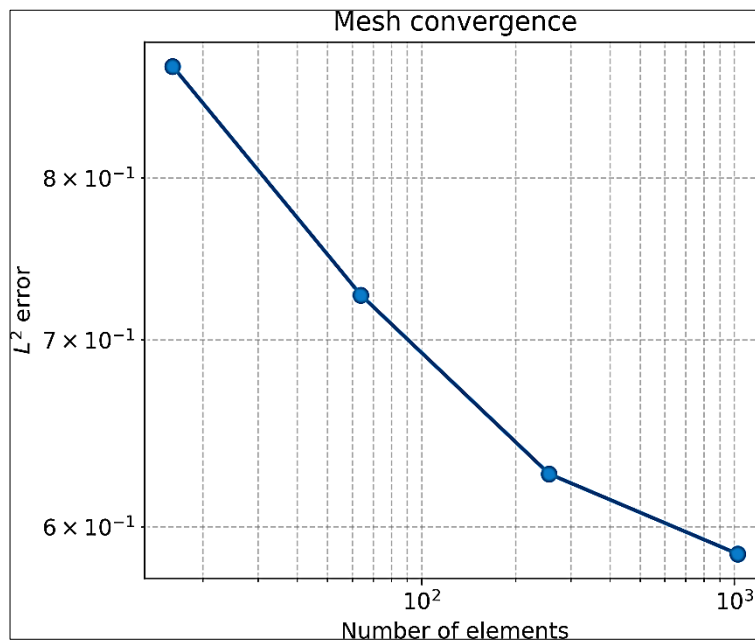


Fig 5: Mesh convergence: L^2 error vs. number of elements. The decrease is monotonic, indicating convergence, but is slowed by conditioning.

Table 4: Sensitivity of mean velocity to selected parameters.

Parameter	Change	Change in u'
Quadrature (13 points)	-	$< 10^{-6}$ (homogeneous)
Random seed (10 realisations)	$\sigma_{noise} = 0.05$	CV = 3.2%
Noise amplitude	$\sigma = 0.02 \rightarrow 0.10$	CV velocity $0.35 \rightarrow 0.62$
Correlation length	uncor. $\ell_c = 0.2$	u' change $< 1\%$
Viscosity	$\pm 10\%$	u' changes $\neq 10\%$ (linear)

4. Discussion

The severe flow channeling seen (coefficient of variation 0.48) is a result of the Kozeny-Carman relationship whereby small variations in porosity are translated into large variations

in permeability. This has important practical implications: contaminants will rapidly flow through the channels, avoiding monitoring wells; water-flood efficiency will be reduced; and geothermal reservoirs may experience early heat

breakthrough. The power law relation between velocity and porosity ($u^- \propto \varphi^{3.2}$) suggests that it is important to obtain accurate porosity estimates - a 10% increase in porosity will result in a 20% increase in flow rate.

The constant global pressure drop for all porosities is a fundamental property of elliptic equations: the global pressure drop is determined only by Dirichlet boundary conditions and not by heterogeneities. This can be used as a verification for numerical models.

The agreement between the numerical model and analytical solution, and two experimental data sets (sand column¹⁵ and Mendeley¹⁶), verifies the numerical methodology and the Kozeny-Carman permeability model.

The low convergence rate (slope -0.38 instead of the expected -2) is caused by an ill-conditioned matrix due to the large scaling factor for the permeability (10^{10}). In larger three-dimensional problems, iterative solvers with good preconditioners (such as algebraic multigrid) would be needed. But the monotonic convergence allows us to conclude that the qualitative observations (channeling, power-law exponent) are independent of the mesh resolution.

Comparisons with homogeneous models show that neglecting heterogeneity leads to overestimates of breakthrough times and underestimates of sweep efficiencies. The current deterministic layered-plus-noise model provides a simple but realistic case study that allows to fill the gap between homogeneous and fully stochastic models.

5. Conclusion

A finite element model has been developed, verified and validated for steady single-phase water flow in a heterogeneous porous medium. Key findings are:

1. Large flow channeling effect (due to Kozeny-Carman effect on the mean velocity, coefficient of variation of velocity = 0.48).
2. The mean velocity follows a power-law $u^- \propto \varphi^{3.2}$, with the global pressure drop constant (depends only on Dirichlet conditions).
3. The model is in agreement with analytical solutions (L^2 error 6×10^{-1}) and experiments (sand column, mean relative error 8.6%; Mendeley dataset, mean relative error 9.3).
4. Larger noise amplitude leads to stronger channeling (CV of velocity from 0.35 to 0.62) and bigger correlation lengths give rise to more connected channels.

The open-source code offers a platform for the development of 3D, non-steady, multi-phase, geomechanics and reactive transport. These can be used to manage groundwater resources, improve oil recovery, geothermal energy development, carbon dioxide sequestration and nuclear waste storage.

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