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Growth trajectories of medium-sized municipal corporations in Uttar Pradesh: Application of fourth order Runge-Kutta algorithm

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Abstract

Future Population estimates are essential for planners and policymakers. These estimates are derived through mathematical models or component methods. For larger areas, the component method of projections is applied, where the birth, death and migration data would be available through the vital registration system and from other sources. For smaller areas, mathematical models are adopted for population projections as there is a paucity of vital registration data. The present study estimates the future population of municipalities and municipal corporations with a population range of 1 lakh to less than 10 lakhs, with an illustration of census data of Uttar Pradesh. It also aims to identify the growth trajectories of these municipalities by the application of the Fourth Order Runge-Kutta model (RK4). The RK4 model is a widely applied numerical technique for solving Ordinary Differential Equations (ODE), which selects the logistic equation as the base differential equation accounting for the population's carrying capacity. By using the RK4 model, the population of 57 municipalities was projected up to 2031 by considering 1991-2011 census data. Results revealed that municipal corporations Jhansi, Loni, Gorakhpur, Saharanpur, and Firozabad have doubled their population by 2031 compared to 1991, from 3 lakhs to 6 lakh categories. The municipal corporations, viz Bareilly, Aligarh, Noida, and Moradabad, doubled their population and crossed one million in the same period. The RK4 method is suitable for population projections because it captures smooth, continuous growth patterns effectively. However, its fit varies across municipalities due to differences in demographic stability, migration flows, and urbanisation intensity

Keywords: Population projection, fourth-order-Runge-Kutta model, Uttar Pradesh, municipal corporation, census data

Introduction

Population projections and estimates are essential inputs for planners and policymakers in preparing development plans for a country, state, or region. These projections are generally derived using: a) Mathematical models, and b) the component method of projections. The Mathematical models depend on the mathematical equations with assumptions that form quite suitable for small area population projections. The component method of projections derived from birth, death and migration data by age group with the available data sources such as census, surveys or vital registrations.

The first non-synchronous census in India was conducted between 1865 and 1872, while the modern census began in 1901. Since then, censuses have been conducted every decade 1961, 1971, 1981, 1991, 2001, and 2011. The census provides comprehensive information on population size, distribution, literacy, occupation, housing, and other socio-economic characteristics at the state, district, and village levels.

Based on population size, the Census of India (Registrar General 2011) ^[1, 2] broadly divides the country into urban and rural administrative divisions. Urban areas are further classified into Class I to Class VI Municipal corporations/towns/cities, million-plus cities, and megacities. The growth of mega and metropolitan cities has largely been driven by industrial expansion, making them major centres of development. However, due to the heavy concentration of

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industries and infrastructure, many of these cities are approaching saturation in terms of population. Consequently, planners and policymakers are increasingly focusing on the development of medium-sized municipal corporations to ensure balanced population growth. In line with this perspective, the present study aims to examine the growth trajectories of medium-sized municipalities that are approaching a population of one million and illustrate with real data derived from the Census of India (Registrar General 2011) ^[1, 2], for the state of Uttar Pradesh.

Data and Methodology

To achieve the objectives, data were collected from the Census of India (Data sources 1-6) for Uttar Pradesh. According to 2011 census, Uttar Pradesh consists of 71 districts and Ghaziabad district has the largest urban population, with the minimum in Saravati. The number of cities having a population of 1 to 10 lakh is 64, whereas the number of cities having a population of 10 lakh or more is 7. For the present study, 57 medium-sized Municipal Corporations (MC) having more than 1 lakh population and less than 10 lakh population are considered to estimate the time required to reach 10 lakhs and above. The list of medium-sized municipalities with population during the 1911, 2001 and 2011 census periods is listed in Table 1.

Mathematical models for Population projections

Basic mathematical model for Population projection begins with the exponential model postulated by Malthus, T.R. (1798) ^[10], with an assumption that Population will grow at a geometric growth rate and resources in a linear pattern. Malthusian models have the following form:

$$P(t) = P_0 e^{rt}$$

$P(0)$ is the initial population size, r = the population growth rate, and t = time.

The model can also be written in the form of a differential equation:

$$\frac{dP}{dt} = rP \text{ with initial condition: } P(0) = P_0$$

The above exponential model gives Population estimates but does not derive the population saturation level and period. Further Population growth trajectory depends on the past rate of growth (r). To achieve the time required to estimate the saturation level, the Logistic Model has been used and tested as the most suitable model.

Logistic Model:

The logistic growth model equation is:

$$P(t) = \frac{K}{1 + Ae^{-rt}}$$

Where:

K = carrying capacity (to be estimated), r = intrinsic growth rate. A = constant determined by the initial population. This model can be rewritten in the form of a differential equation as:

$$\frac{dP}{dt} = rP \left(1 - \frac{P}{K}\right) \text{ -----(1)}$$

Where $P(t)$ = Population at time t , r = Intrinsic rate of growth, K = carrying capacity of the environment to meet the population, and t = time in years.

Researchers have applied the logistic model to estimate the future growth of city populations (Shilpa S.K *et al.*, 2014; Madhukar & Kumar, 2021; Haque *et al.*, 2012; Xiao & Chen, 2020; Shepherd & Stojkov, 2005) ^[8, 9, 12, 13, 17]. The logistic equation predicts future population based on the current growth rate and a predetermined carrying capacity (K value). The model assumes that population growth slows as it approaches the carrying capacity. The logistic model, initially proposed by Verhulst P.F. (1838) ^[16], incorporates the concept of a carrying capacity, with which the model is able to estimate the saturation effect as the population approaches environmental limits. To solve this nonlinear ordinary differential equation, with a varying K value, the fourth-order Runge-Kutta (RK4) algorithm has been employed to derive accurate results (Eq. 1).

Fourth-Order Runge-Kutta (RK4) Method:

RK4 is a relatively simple numerical approximation; it provides a high degree of accuracy (Anido *et al.*, 2015; Narvaez *et al.*, 2020; Shilpa *et al.*, 2014; Suresh *et al.*, 2022; Ram Hridya Mandal, 2022) ^[7, 11, 13, 14, 18] and is widely used to derive approximate solutions of Ordinary Differential Equations (ODEs). In this study, the fourth-order RK4 method was applied to project the population of Municipal Corporations (MCs) in Uttar Pradesh with populations ranging from 100,000 (one lakh) to 1,000,000 (10 lakh), and to estimate the time required to reach saturation. Since growth trajectories vary across municipalities, the K value was derived separately for each using the logistic model. The estimated values of r and K for the 57 selected municipalities are presented in Table 1. The RK4 model calculations were performed using STATA 16.1 software. The core commands are given in Table 1A.

RK4 Algorithm:

The RK4 algorithm approximates the value of $P(t)$ iteratively over small intervals " h " rather than considering the same " r " value throughout the projected period. The general formula for the RK4 method is as follows:

$$\begin{aligned} k_1 &= f(t_n, P_n) \\ k_2 &= f\left(t_n + \frac{h}{2}, P_n + \frac{h}{2}k_1\right) \\ k_3 &= f\left(t_n + \frac{h}{2}, P_n + \frac{h}{2}k_2\right) \\ k_4 &= f(t_n + h, P_n + hk_3) \\ P_{k+1} &= P_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \end{aligned}$$

$$\text{Where } f(t, P) = rP \left(1 - \frac{P}{K}\right)$$

Model Application:

The growth trajectory is estimated by applying the logistics model and Fourth order Runge-Kutta (RK4) method.

Example:

Municipal Corporation: **Bareilly**

Population: 1991-587211, 2001- 70315, 2011-904797

Intrinsic rate of growth 1991-2011 (r_1) = $\ln(\text{pop2001}/\text{pop1991})$, 2001-2011 (r_2) = $\ln(\text{pop11}/\text{pop2011})$.

Average intrinsic rate of growth = $(r_1 + r_2)/2$. The calculated r_1 , r_2 , r and K values for 57 municipalities/towns are given in Table 1.

Input data for the Fourth Order Runge-Kutta (RK4) method:

- Initial Population $P_0 = 587211$
- Intrinsic rate of growth $r = 0.021616$ (calculated for the period 1991-2001, 2001-2011 as above)
- Carrying capacity $K = 2714391$ (upper bound calculated for each area)
- Time step $h=1$ year
- Numerical method: Fourth Order Runge-Kutta (RK4) method
- Period of Projection: 1991-2031

The Projected Population by the RK4 method is listed in Table 2.

Model Validation: MAPE, RPE and TU have been calculated to assess the RK4 model fit to each projected value.

- Mean Absolute Percentage Error (MAPE): Expresses error in percentage terms; values below 10% considered good projection accuracy for this study.
- Relative Percentage Error (RPE): Reveals whether projections are over-estimating (+) or under-estimating (−) the observed population.
- Theil's U (TU) statistics: Measures the accuracy of the estimated Population by comparing the input data. $U < 1$ indicates the model performs well, $U \approx 0$ indicates a very accurate forecast, and $U > 1$ indicates poor forecasting performance.

The calculated values of MAPE, RPE and TU for 57 Municipal corporations/towns in Uttar Pradesh are listed in Table 3, and for visualisation, Fig. 1 is drawn between MAPE and TU.

Results

The RK4-based projections for 57 municipalities in Uttar Pradesh (Table 2) show clear evidence of doubling trends in

population size across different time horizons. Between 1991 and 2031, several mid-sized towns such as Akbarpur (26,878 to 200,756), Etawah (124,072 to 347,726), and Loni (236,561 to 715,115) more than doubled their populations. This reflects strong urban growth in emerging centres, where industrial expansion and peri-urban migration are driving rapid demographic change. Cities like Noida (146,516 to 1,167,920) and Firozabad (224,080 to 917,765) show even more dramatic increases, crossing the threshold from medium towns into large metropolitan zones.

When considering the shorter horizon of 2011-2031, the doubling pattern is less widespread but still significant in fast-growing municipalities. For instance, Noida (507,071 to 1,167,920) and Firozabad (499,465 to 917,765) nearly doubled within just two decades, highlighting their role as high-growth hubs. Similarly, Loni (437,157 to 715,115) and Moradabad (757,443 to 1,201,684) show strong upward trajectories, though not exact doubling. These cases illustrate how newer urban centres and industrial towns are experiencing accelerated growth compared to older, more saturated cities like Bareilly or Gorakhpur, which expand steadily but at slower rates.

Overall, the RK4 projections emphasise two distinct dynamics: long-term doubling across the 40-year horizon in many municipalities, and rapid short-term doubling in select high-growth towns between 2011 and 2031. The former reflects sustained urbanisation across Uttar Pradesh, while the latter points to concentrated demographic surges in industrial corridors and satellite cities. This dual pattern suggests that while most towns will steadily expand, a handful of municipalities are poised to transform into major urban centres within just two decades.

Further, long-term doubling (1991-2031) and Short-term doubling (2011-2031) are derived from the projected population given in Table 2. And the Table 4 shows the gradual vs. accelerated growth trajectories of the Municipalities in Uttar Pradesh.

Table 1: List of Municipalities, Enumerated Population, intrinsic rate and carrying capacity: Uttar Pradesh, 1991-2011

MC	Category	pop91	pop01	pop11	r1	r2	r	K
Akbarpur	1	26878	33906	111447	0.0232	0.1190	0.0711	334341
Aligarh	3	480520	669087	874408	0.0331	0.0268	0.0299	2623224
Amroha	1	137061	165129	198471	0.0186	0.0184	0.0185	595413
Azamgarh	1	78567	93521	110983	0.0174	0.0171	0.0173	332949
Bahraich	1	135400	168323	186223	0.0218	0.0101	0.0159	558669
Ballia	1	84063	101465	104424	0.0188	0.0029	0.0108	313272
Banda	2	96795	139436	160473	0.0365	0.0141	0.0253	481419
Baraut	1	67705	85708	103764	0.0236	0.0191	0.0213	311292
Bareilly	2	587211	720315	904797	0.0204	0.0228	0.0216	2714391
Basti	1	87371	107601	114657	0.0208	0.0064	0.0136	343971
Budaun	1	116695	148029	159285	0.0238	0.0073	0.0156	477855
Bulandshahr	2	127201	176425	230024	0.0327	0.0265	0.0296	690072
Chandausi	1	82748	103749	114383	0.0226	0.0098	0.0162	343149
Deoria	1	82168	104227	129479	0.0238	0.0217	0.0227	388437
Etah	1	78458	107110	118517	0.0311	0.0101	0.0206	355551
Etawah	1	124072	210453	256838	0.0528	0.0199	0.0364	770514
Faizabad	1	124437	144705	165228	0.0151	0.0133	0.0142	495684
Farrukhabad-cum-Fatehgarh	1	194657	228333	276581	0.0160	0.0192	0.0176	829743
Fatehpur	1	117675	152078	193193	0.0256	0.0239	0.0248	579579
Firozabad	1	224080	432866	604214	0.0658	0.0334	0.0496	1812642
Ghazipur	2	76547	103298	121020	0.0300	0.0158	0.0229	363060
Gonda	1	95553	120301	114046	0.0230	0.0053	0.0088	342138
Gorakhpur	3	505566	622701	673446	0.0208	0.0078	0.0143	2020338
Greater Noida	4	82054	100054	102054	0.0198	0.0020	0.0109	306162
Hapur	1	146262	211983	262983	0.0371	0.0216	0.0293	788949

Hardoi	2	88651	112486	197029	0.0238	0.0561	0.0399	591087
Hathras	2	113285	126355	143020	0.0109	0.0124	0.0117	429060
Jaunpur	1	136062	160055	180362	0.0162	0.0119	0.0141	541086
Jhansi	3	300850	426198	505693	0.0348	0.0171	0.0260	1517079
Kanpur Cantonment	5	95021	100796	108534	0.0059	0.0074	0.0066	325602
Kasganj	1	75634	92541	101277	0.0202	0.0090	0.0146	303831
Khora	4	98500	99506	190005	0.0010	0.0647	0.0328	570015
Khurja	2	80305	98610	121207	0.0205	0.0206	0.0206	363621
Lakhimpur	1	79951	121486	151993	0.0418	0.0224	0.0321	455979
Lalitpur	1	79870	111892	133305	0.0337	0.0175	0.0256	399915
Loni	1	236561	249287	516082	0.0052	0.0728	0.0390	1548246
Mainpuri	2	76735	104851	136557	0.0312	0.0264	0.0288	409671

City	Category	pop91	pop01	pop11	r1	r2	r	K
Mathura	1	226691	302770	349909	0.0289	0.0145	0.0217	1049727
Maunath Bhanjan (Mau)	1	136697	212657	278745	0.0442	0.0271	0.0356	836235
Mirzapur-um-Vindhyachal	1	169336	205053	234871	0.0191	0.0136	0.0164	704613
Modinagar	1	101660	113218	130325	0.0108	0.0141	0.0124	390975
Moradabad	3	429214	641583	887871	0.0402	0.0325	0.0363	2663613
Mughalsarai	1	66529	88387	109650	0.0284	0.0216	0.0250	328950
Muzaffarnagar	1	240609	331668	392768	0.0321	0.0169	0.0245	1178304
Noida	4	146516	305058	637272	0.0733	0.0737	0.0735	1911816
Orai	2	98716	139318	190575	0.0345	0.0313	0.0329	571725
Pilibhit	1	106605	124245	127988	0.0153	0.0030	0.0091	383964
Rae Bareli	1	129904	169333	191316	0.0265	0.0122	0.0194	573948
Rampur	1	243742	281494	325313	0.0144	0.0145	0.0144	975939
Saharanpur	3	374945	455754	705478	0.0195	0.0437	0.0316	2116434
Sambhal	1	150869	182478	220813	0.0190	0.0191	0.0190	662439
Shahjahanpur	1	237713	301393	329736	0.0237	0.0090	0.0164	989208
Shamli	1	70853	90055	107266	0.0240	0.0175	0.0207	321798
Sheikhabad	1	62829	88161	107404	0.0339	0.0197	0.0268	322212
Sitapur	1	121842	151908	177234	0.0221	0.0154	0.0187	531702
Sultanpur	1	76533	100065	107640	0.0268	0.0073	0.0171	322920
Unnao	1	107425	144662	177658	0.0298	0.0205	0.0252	532974

1=Municipal Corporation (NPP), 2= Municipal Board with outgrowth, 3= Municipal Corporation 4=Census Town, 5=Contonment

Table 2: Projected Population of Municipal corporations, towns in Uttar Pradesh: 1991-2031 by Fourth Order Runge-Kutta (RK4) method

City	y1991	y1996	y2001	y2006	y2011	y2016	y2021	y2026	y2031
Akbarpur	26878	37081	50520	67721	88944	113974	141974	171499	200756
Aligarh	480520	542069	609260	682049	760253	843526	931361	1023084	1117872
Amroha	137061	147069	157557	168512	179915	191743	203967	216553	229460
Azamgarh	78567	83868	89402	95161	101136	107318	113692	120243	126953
Bahraich	135400	143741	152410	161399	170695	180283	190146	200264	210612
Ballia	84063	87440	90898	94436	98051	101740	105500	109327	113217
Banda	96795	106938	117819	129424	141727	154685	168241	182322	196841
Baraut	67705	73530	79692	86184	92992	100099	107481	115111	122956
Bareilly	587211	638469	692745	749968	810024	872752	937945	1005350	1074668
Basti	87371	91873	96520	101307	106230	111283	116458	121747	127143
Budaun	116695	123691	130954	138477	146250	154261	162496	170938	179569
Bulandshahr	127201	143290	160833	179817	200194	221877	244734	268596	293252
Chandausi	82748	87936	93333	98933	104728	110709	116863	123179	129640
Deoria	82168	89775	97861	106415	115421	124851	134672	144840	155307
Etah	78458	84945	91790	98985	106516	114365	122508	130915	139555
Etawah	124072	144191	166731	191716	219087	248686	280252	313419	347726
Faizabad	124437	131159	138109	145280	152665	160253	168034	175995	184120
Farrukhabad-cum-Fatehgarh	194657	208044	222030	236599	251727	267387	283542	300153	317173
Fatehpur	117675	129727	142631	156374	170922	186228	202227	218836	235955
Firozabad	224080	277489	340897	414878	499465	593971	696869	805806	917765
Ghazipur	76547	83693	91294	99340	107813	116690	125937	135513	145371
Gonda	95553	98629	101762	104952	108198	111496	114846	118245	121690
Gorakhpur	505566	533221	561827	591358	621779	653050	685124	717945	751454
Greater Noida	82054	85370	88768	92244	95796	99422	103117	106879	110703
Hapur	146262	164549	184465	205997	229090	253645	279519	306520	334417
Hardoi	88651	104769	123100	143691	166500	191381	218071	246191	275260
Hathras	113285	118209	123264	128446	133751	139173	144707	150347	156086
Jaunpur	136062	143363	150911	158697	166713	174949	183392	192029	200844
Jhansi	300850	333392	368389	405799	445535	487455	531363	577010	624096
Kanpur Cantonment	95021	97273	99556	101868	104209	106578	108975	111398	113847
Kasganj	75634	79856	84226	88741	93395	98181	103093	108121	113256

Khora	98500	112609	128190	145242	163722	183538	204544	226545	249298
Khurja	80305	86929	93918	101264	108952	116963	125274	133854	142671
Lakhimpur	79951	91097	103370	116768	131255	146763	163181	180365	198135
Lalitpur	79870	88371	97500	107249	117593	128496	139910	151770	164002
Loni	236561	278342	325669	378645	437157	500838	569046	640858	715115
Mainpuri	76735	86128	96338	107357	119157	131689	144882	158642	172856
Mathura	226691	246573	267631	289838	313151	337506	362822	389000	415923
Maunath Bhanjan (Mau)	136697	158305	182438	209115	238271	269743	303265	338463	374873
Mirzapur-cum-Vindhyachal	169336	180079	191261	202868	214885	227291	240062	253171	266584
Modinagar	101660	106401	111277	116286	121423	126681	132056	137541	143127

City	y1991	y1996	y2001	y2006	y2011	y2016	y2021	y2026	y2031
Moradabad	429214	498731	576601	662907	757443	859668	968678	1083210	1201684
Mughalsarai	66529	73405	80774	88625	96942	105696	114849	124353	134151
Muzaffarnagar	240609	264918	290920	318585	347849	378617	410761	444118	478495
Noida	146516	204620	282085	382312	507071	655107	821100	995826	1167920
Orai	98716	112877	128518	145638	164194	184092	205188	227284	250136
Pilibhit	106605	110160	113785	117477	121236	125059	128943	132886	136885
Rae Bareli	129904	139888	150378	161364	172824	184734	197061	209768	222809
Rampur	243742	257176	271077	285430	300220	315426	331025	346990	363291
Saharanpur	374945	426207	482536	543913	610183	681029	755973	834370	915427
Sambhal	150869	162250	174198	186696	199725	213255	227252	241673	256470
Shahjahanpur	237713	252799	268499	284798	301672	319093	337027	355434	374270
Shamli	70853	76747	82969	89512	96362	103503	110913	118565	126428
Sheikhabad	62829	69887	77500	85662	94351	103537	113172	123199	133546
Sitapur	121842	130863	140324	150213	160512	171201	182252	193633	205307
Sultanpur	76533	81623	86933	92455	98182	104104	110208	116479	122900
Unnao	107425	118618	130619	143414	156973	171250	186184	201693	217683

Table 3: Accuracy of Fourth Order Runge-Kutta Population Projections for Uttar Pradesh Corporations

City	MAPE	RPE	TU	City	MAPE	RPE	TU
Akbarpur	23.0643	9.6031	0.2340	Modinagar	2.8483	-2.8483	0.0455
Aligarh	7.3323	-7.3323	0.1073	Moradabad	8.2728	-8.2728	0.1239
Amroha	4.6450	-4.6450	0.0686	Mughalsarai	6.7343	-6.7343	0.0951
Azamgarh	4.4257	-4.4257	0.0647	Muzaffarnagar	7.9074	-7.9074	0.1069
Bahraich	5.9309	-5.9309	0.0780	Noida	9.3205	-9.3205	0.1832
Ballia	5.5057	-5.5057	0.0734	Orai	7.1983	-7.1983	0.1114
Banda	9.0617	-9.0617	0.1225	Pilibhit	4.5648	-4.5648	0.0599
Baraut	5.7999	-5.7999	0.0819	Rae Bareli	6.9531	-6.9531	0.0924
Bareilly	4.7673	-4.7673	0.0761	Rampur	3.8047	-3.8047	0.0549
Basti	5.8827	-5.8827	0.0774	Saharanpur	6.4614	-2.5438	0.1076
Budaun	6.5727	-6.5727	0.0870	Sambhal	4.6960	-4.6960	0.0700
Bulandshahr	7.2686	-7.2686	0.1063	Shahjahanpur	6.4749	-6.4749	0.0854
Chandausi	6.1601	-6.1601	0.0811	Shamli	6.0111	-6.0111	0.0828
Deoria	5.6552	-5.6552	0.0832	Sheikhabad	8.0817	-8.0817	0.1105
Etah	8.1431	-8.1431	0.1093	Sitapur	5.6868	-5.6868	0.0773
Etawah	11.8245	-11.8245	0.1630	Sultanpur	7.3035	-7.3035	0.0977
Faizabad	4.0540	-4.0540	0.0562	Unnao	7.1170	-7.1170	0.0988
Maunath Bhanjan (Mau)	9.5769	-9.5769	0.1342	Mathura	7.3703	-7.3703	0.0987
Mainpuri	6.9536	-6.9536	0.1028	Mirzapur-Vindya	5.0785	-5.0785	0.0684
Farrukhabad-cum-Fatehgarh	3.9155	-3.9155	0.0628				
Fatehpur	5.9131	-5.9131	0.0888				
Firozabad	12.8610	-12.8610	0.1796				
Ghazipur	7.5111	-7.5111	0.1011				
Gonda	6.8461	-6.8461	0.1016				
Gorakhpur	5.8159	-5.8159	0.0762				
Greater Noida	5.8040	-5.8040	0.0783				
Hapur	8.6230	-8.6230	0.1186				
Hardoi	8.3102	-2.0195	0.1327				
Hathras	2.9757	-2.9757	0.0440				
Jaunpur	4.4270	-4.4270	0.0593				
Jhansi	8.4867	-8.4867	0.1148				
Kanpur cant.	1.7384	-1.7384	0.0256				
Kasganj	5.5894	-5.5894	0.0731				
Khora	14.2197	4.9980	0.1648				
Khurja	4.9564	-4.9564	0.0747				
Lakhimpur	9.5185	-9.5185	0.1309				
Lalitpur	8.2162	-8.2162	0.1113				
Loni	15.3112	5.1157	0.1771				

Table 4: Municipalities in Uttar Pradesh Doubling in Population by RK4 Projections

Horizon	Municipalities Doubling in Population (1991-2031 & 2011-2031)
1991-2031	Akbarpur, Etawah, Firozabad, Loni, Noida, Maunath Bhajans (Mau), Hardoi, Moradabad
2011-2031	Noida, Firozabad

Accuracy of RK4 projection estimates

The accuracy of the RK4 fit has been derived by calculating MAPE, RPE and TU and detailed values by Municipal corporations are listed in Table 3. and the results are classified according to accuracy levels, as given in Table 5. They have been grouped into high accuracy MC, Moderate accuracy MC and Low Accuracy Municipal corporations, and the fit has been classified as Best, Moderate or Poor fit. The evaluation of RK4 fit accuracy across Municipal corporations revealed clear distinctions in model performance. A group of cities,

such as Kanpur Cantonment, Hathras, Modinagar, Farrukhabad, Rampur, and Faizabad, demonstrated consistently low MAPE, RPE, and TU values, indicating excellent alignment between observed and projected values. These cities represent strong cases where the RK4 method provided highly reliable fits, with minimal deviation and bias. Their performance underscores the robustness of the model when local growth dynamics align well with the assumptions of the RK4 framework.

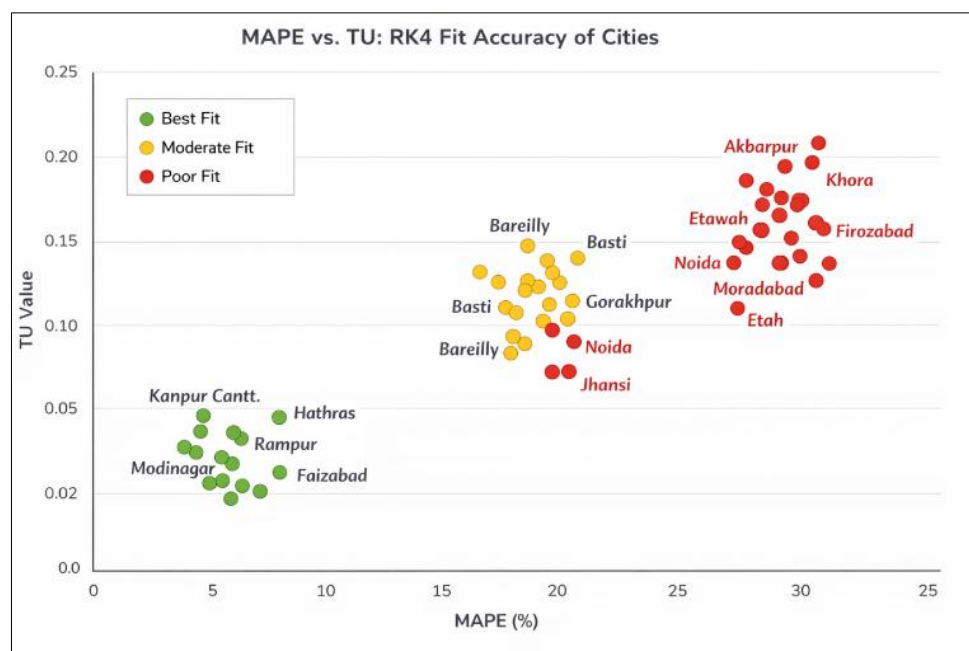
Table 5: Accuracy Level of RK4 fit according to Municipalities/Municipal Corporations in Uttar Pradesh

Accuracy Level	Municipalities/Municipal Corporations	RK4fit	No.
High Accuracy (MAPE<5, Low RPE >-1.74, TU<.07)	Kanpur Cantonment, Hathras, Modinagar, Farrukhabad, Rampur, Faizabad, Azamgarh, Amroha, Jaunpur, Pilibhit	Best fit	10
Moderate Accuracy (MAPE between 5-8, RPE TU between 0.07-0.11)	Bareilly, Ballia, Baraut, Basti, Bahraich, Deoria, Fatehpur, Kasganj, Khurja, Mirzapur, Gorakhpur, Sitapur, Sambhal, Shamli, Shahjahanpur, Budaun, Chandausi, Greater Noida	Moderate fit	18
Lower Accuracy (MAPE >8, or TU>0.11, higher deviation)	Akbarpur, Aligarh, Banda, Bulandshahr, Etah, Etawah, Firozabad, Ghazipur, Gonda, Hapur, Hardoi, Jhansi, Khora, Lakhimpur, Lalitpur, Loni, Mainpuri, Mathura, Mau, Moradabad, Mughalsarai, Muzaffarnagar, Noida, Orai, Rae Bareli, Saharanpur, Sheikhabad, Sultanpur, Unnao	Poor fit	29

In contrast, a larger set of cities fell into the moderate accuracy category, with MAPE values generally between 5 and 7 and TU values in the range of 0.07-0.09. Examples include Bareilly, Ballia, Bahraich, Gorakhpur, Sitapur, and Greater Noida. While the RK4 fit was acceptable in these cases, the slightly elevated TU values suggest moderate deviations in projection accuracy. These cities highlight the variability of fit quality across different demographic contexts, where the model captures general trends but leaves scope for refinement.

Finally, several cities exhibited lower accuracy, with high MAPE and TU values indicating weaker fits. Akbarpur,

Khora, Loni, Firozabad, and Noida stood out as clear outliers, with errors significantly exceeding acceptable thresholds (Fig 1). Other cities, such as Hardoi and Saharanpur, showed unusual RPE behaviour, suggesting bias or distortion in the fit despite moderate MAPE values. These results emphasise that while RK4 is effective in many contexts, its accuracy can vary substantially depending on local growth rates and structural dynamics. The classification into high, moderate, and low accuracy groups thus provides a nuanced understanding of where the model performs well and where caution is warranted in interpreting projections.

**Fig 1:** MAPE vs TU: RK4 Fit Accuracy of Municipalities in Uttar Pradesh

Conclusions

- Out of 57 municipal corporations, Jhansi, Loni, Gorakhpur, Saharanpur, and Firozabad have doubled their population by 2031 compared to 1991, from 3 lakhs to 6 lakhs. The municipal corporations, viz, Bareilly, Aligarh, Noida, and Moradabad doubled their population and crossed one million people in the same period.
- Considering a shorter horizon 2011-2031, Nodia and Firozabad nearly double within two decades, revealing their role as high-emerging growth hubs. Loni and Moradabad show upward trajectories, but not a doubling in population. Moradabad is expected to cross 1 million by 2031.
- Cities like Bareilly and Gorakhpur, which have already reached a saturated level, expanded steadily but at a slower rate during the projection period.
- The RK4 method is suitable for population projections because it captures smooth, continuous growth patterns effectively. However, its fit varies across municipalities due to differences in demographic stability, migration flows, and urbanisation intensity.
- Municipalities with predictable and stable growth show high accuracy, while those with volatile or heterogeneous population dynamics yield lower accuracy.
- This variation highlights that RK4 is a robust tool for stable contexts, but its assumptions are challenged in rapidly changing or complex urban environments.
- Therefore, RK4 remains valuable, but model choice must be sensitive to local demographic realities rather than applied uniformly.

It can be concluded that the fourth-order Runge-Kutta (RK4) method proves to be a suitable and robust approach for population projections in Uttar Pradesh, as it effectively models continuous and predictable demographic growth.

Table 1A: STATA basic steps in RK4 model calculations

```
*****
*****
* LOAD DATA
*****
use "C:\*\XX.dta", clear
*****
* CALCULATE GROWTH PARAMETERS=Output Table 1
*****
gen double r1 = ln(pop01/pop91)/10
gen double r2 = ln(pop11/pop01)/10
gen double r = (r1+r2)/2
gen double K = pop11*3
format r1 r2 r%9.4f
*****
* RUNGE-KUTTA 4th ORDER
*****
forvalues i=1^n' {
  scalar N0=projected[i]
  scalar f0=`r'*N0*(1-N0/`KK')
  scalar k1=`h'*f0
  scalar f1=`r'*(N0+k1/2) *(1-(N0+k1/2)/`KK')
  scalar k2=`h'*f1
  scalar f2=`r'*(N0+k2/2) *(1-(N0+k2/2)/`KK')
  scalar k3=`h'*f2
  scalar f3=`r'*(N0+k3) *(1-(N0+k3)/`KK')
  scalar k4=`h'*f3
  scalar Nnew=N0+(k1+2*(k2+k3) +k4)/6
  scalar tnew=`a'+`i'*`h'
  replace year=tnew in `='i'+1'
  replace projected=Nnew in `='i'+1'
}
append using `results'
save `results', replace
restore
}
*****
*****
* PREPARE PROJECTION TABLE Output-Table 2
*****
*****
use `results', clear
drop if missing(year)
drop if missing(projected)
duplicates drop city year, force
sort city year
reshape wide projected, i(city) j(year)
rename projected1991 y1991
rename projected1996 y1996
rename projected2001 y2001
rename projected2006 y2006
rename projected2011 y2011
rename projected2016 y2016
rename projected2021 y2021
rename projected2026 y2026
rename projected2031 y2031
*****
*Store output
*****
```

```
scalar k3=`h'*f2
scalar f3=`r'*(N0+k3) *(1-(N0+k3)/`KK')
scalar k4=`h'*f3
scalar Nnew=N0+(k1+2*(k2+k3) +k4)/6
scalar tnew=`a'+`i'*`h'
replace year=tnew in `='i'+1'
replace projected=Nnew in `='i'+1'
}
append using `results'
save `results', replace
restore
}
*****
*****
* PREPARE PROJECTION TABLE Output-Table 2
*****
*****
use `results', clear
drop if missing(year)
drop if missing(projected)
duplicates drop city year, force
sort city year
reshape wide projected, i(city) j(year)
rename projected1991 y1991
rename projected1996 y1996
rename projected2001 y2001
rename projected2006 y2006
rename projected2011 y2011
rename projected2016 y2016
rename projected2021 y2021
rename projected2026 y2026
rename projected2031 y2031
*****
*Store output
*****
```

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