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Extensions of compatibility concepts in 2-metric spaces with applications to four-map fixed point theorems

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Abstract

Within the context of 2-metric spaces, this work presents two novel ideas: sub sequential continuity and sub compatibility. A more general framework for studying fixed point theory is provided by demonstrating that these ideas are less robust than the current ones of reciprocal continuity and occasionally weak compatibility. Using these recently proved conditions, we prove a shared fixed-point theorem for four 2-metric self-maps. This study's findings generalise and extend a number of famous findings from other works, such as those of Bouhadjera and Godet-Thobie as well as Jungck and Rhoades. In addition, our results enhance and harmonise a number of preexisting concepts, including commuting, weakly commuting, compatible, semi-compatible, and occasionally weakly compatible mappings. The suggested method's generalisability shows that it could be useful in expanding nonlinear analysis and fixed-point theory.

Keywords: Sub compatibility, sub sequential continuity, metric spaces, fixed point theorem, nonlinear analysis

Introduction

Nonlinear analysis has traditionally revolved on fixed point theory because of its pervasive relevance in the hard and soft sciences, as well as in engineering and mathematics. Fixed points in metric spaces are real and unique because of classical results like Banach's Contraction Principle. Traditional metric conditions are inadequate for handling complicated systems, although academics have expanded these principles to more generalised structures over time. The idea of 2-metric spaces is a major generalisation in this regard; in these spaces, distance is defined not between two points but among three points at the same time. Area, volume, and higher-dimensional interaction problems are some of the most challenging to model, but this framework offers a more robust geometric structure and makes it possible.

In the framework of fixed-point theory, compatibility notions are essential for guaranteeing that common fixed points exist for various mappings. From its inception as a means to reduce commutativity requirements, compatibility has evolved into other variants, such as semi-compatibility, weak compatibility, and even weakly compatible mappings. Mathematicians have been able to prove fixed point results under less stringent conditions thanks to these generalisations. Due to the triadic character of the distance function, the behaviour of mappings in 2-metric spaces is intrinsically more complex, which poses new challenges and opportunities when these compatibility principles are extended to such spaces.

In the context of many mappings, the extension of compatibility ideas in 2-metric spaces becomes crucial. Multiple interacting functions, rather than a single transformation, govern systems in many real-world contexts. As a result, two, three, or even more mappings shared fixed points are investigated. The most general and complicated of these are four-map fixed point theorems. Four self-maps constructed on a 2-metric space can share a fixed point under certain conditions, which are studied in these theorems. Compatibility and continuity assumptions are typically complex.

Further, strict commutativity and continuity restrictions can be relaxed with the introduction of generalised compatibility conditions in 2-metric spaces.

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Because of this, the theory can be applied more effectively to situations in the real world, where perfect conditions are seldom present. For example, in optimisation problems, computational models, and dynamical systems, mappings may not commute precisely but nevertheless show convergence behaviour that can be described by weaker compatibility definitions. As a result, a more adaptable and practical analysis framework is provided by expanding compatibility principles.

The creation of novel contraction-type conditions appropriate to 2-metric spaces is another significant part of this research. In 2-metric spaces, traditional pairwise distance-based contraction mappings need to be recast as three-variable functions. Such generalised contractions, when coupled with expanded compatibility requirements, provide robust fixed-point outcomes. These findings not only bring together different theorems that already exist, but also pave the way for the discovery of new ones, especially for four mappings where interactions get very complicated.

Finally, a major step forward in contemporary fixed-point theory has been the generalisation of compatibility theorems in 2-metric spaces to four-map fixed point theorems. It improves theoretical understanding and practical applicability by expanding the field of analysis to include more generalised structures and weaker circumstances. This is an important and dynamic topic of study because it has the ability to solve complicated issues in mathematics and related fields, which keeps drawing researchers' interest.

Review of literature

Karn, Uday. (2025) ^[21]. A α -compatible mapping of type (P) constructed by an appropriate control function is introduced in this paper. Using this novel idea, we prove a well-known fixed-point theorem in a full metric space. The acquired outcome expands upon a number of relevant findings that have been published in the literature, while also generalising the theorem of previous researcher.

Pathak, Renu & Gore, Rohini. (2025) ^[13]. A common fixed-point theorem for two weakly compatible mappings in Fuzzy PMS matching the E.A. property is established in this study. We expand upon the current Fuzzy PMS with a novel idea of E.A. Fuzzy Partial Metric spaces. Two weakly compatible mappings in E.A. Fuzzy Partial Metric spaces can have a common fixed point, according to our primary finding. The findings are highly relevant to fuzzy topology, fixed-point theory, and optimisation issues with uncertain or imprecise data as well as to image processing, machine learning, and related fields.

Goyal, A.K. (2018) ^[6]. The fixed-point theorems for six mappings using rational contractive conditions in full metric spaces are proven. utilising the concepts of compatibility, weak compatibility, and commutativity. A common fixed-point theorem for six mappings is proved in this study utilising the concept of compatible mapping of type (A).

Tran Van *et al.*, (2014) ^[19]. Theorems on fixed points in metric and 2-metric spaces are introduced in this study in a novel way that generalises and integrates numerous previous findings. To demonstrate the outcomes, examples are provided.

Singh, Bijendra & Jain, Shishir. (2005) ^[18]. In order to prove a fixed-point theorem for four self-maps, this study aims to provide the idea of two semi-compatible self-maps in a fuzzy metric space. It provides an expansion of Va-suki [10] to four self-maps when compatibility and semi-compatibility are assumed, respectively. Additionally, these findings corroborate the contrasting conclusions reached by previous authors.

Naidu, S. (2001) ^[17]. Solving specific generalised contraction conditions, we prove fixed point theorems for pairs and quadruples of self-maps on a 2-metric space, and we present the idea of compatibility for these pairs. Additional metric space variants of the same have also been found.

Preliminaries

Definition 1. The following conditions must be satisfied for a non-empty set X to have a real-valued function d on $X \times X \times X$:

1. $d(x, y, z) = 0$ if at least two of x, y, z , are identical,
2. $d(x, y, z) = d(p(x, y, z))$ for all $x, y, z \in X$ and each permutation $p(x, y, z)$ of x, y, z ,
3. $d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z)$ for all $x, y, z, w, X \in$.

A 2-metric function d on X and the corresponding 2-metric space are both defined by the notation (X, d) .

Definition 2. Two self-mappings A and S of a 2-metric space (X, d) are considered compatible if, for all $a \in X$, where $\{a_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} a_n = a$, the limit of the set $(ASx_n, SAsx_n, a)$ is equal to $\lim_{n \rightarrow \infty} a_n$. For all x in X , the product of AS_n and Sx_n is equal to $\{x_n\}$.

Definition 3. Weakly compatible self-maps A and S of a 2-metric space (X, d) are those that commute at their coincidence points, meaning that for any $x \in X$, $SAx = ASx$.

Definition 4. For all $a \in X$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} x_n = a$, two self mappings $A, S: X \rightarrow X$ are considered semi-compatible if $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = a$. all values of x where X is an element of

Here we illustrate the connection between compatible, weakly compatible, and semi-compatible maps with a few instances.

Results of the study

Our primary finding is now demonstrated.

Theorem 1. Suppose there are four self-maps on a two-metric space (X, d) : f, g, h , and k . First, there is a coincidence point between f and h , and second, there is a coincidence point between g and k , provided that (g, k) and (f, h) are sub compatible and sub sequentially continuous.

In addition, in the event that

$$d(fx, gy, a) \leq \phi \left[\max \left\{ \begin{array}{l} d(hx, ky, a), d(fx, hx, a), d(gy, ky, a) \\ d(hx, gy, a), d(ky, fx, a) \end{array} \right\} \right] \text{ for all}$$

For each value of 0 that is less than or equal to s, the continuous function $\phi: [0,1] \rightarrow [0,1]$ is greater than s, and the set X includes x and y. It follows that f, g, h, and k all share a single fixed point in X.

Proof. Since the pairs (f, h) and (g, k) are sub compatible and sub sequentially continuous, therefore, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_n fx_n, hx_n = u$ for some $u \in X$ and which satisfy $\lim_n (fx_n, hx_n, a) = d(fu, hu, a) = 0$, $\lim_n gy_n, ky_n = v$ for some $v \in X$ and which satisfy $\lim_n (gy_n, ky_n, a) = d(gv, kv, a) = 0$.

Thus, u is the point of coincidence for f and h, and v is the point of coincidence for g and k. Similarly, $gv = kv$ and $fu = hu$.

First, we get by plugging x_n and y_n into (1).

$$d(fx_n, gy_n, a) \leq \phi \left[\max \left\{ \begin{array}{l} d(hx_n, ky_n, a), d(fx_n, hx_n, a), d(gy_n, ky_n, a) \\ d(hx_n, gy_n, a), d(ky_n, fx_n, a) \end{array} \right\} \right]$$

Although n approaches infinity,

$$d(u, v, a) \leq \phi \left[\max \{d(u, v, a), 0, 0, d(u, v, a), d(u, v, a)\} \right]$$

$$\text{i.e., } d(u, v, a) \leq \phi[d(u, v, a)] > d(u, v, a), \text{ a contradiction.}$$

Therefore, u multiplied by v equals.

Then, for $x = u$ and y_n , we get back to (1) and get

$$d(fu, gy_n, a) \leq \phi \left[\max \left\{ \begin{array}{l} d(hu, ky_n, a), d(fu, hu, a), d(gy_n, ky_n, a) \\ d(hu, gy_n, a), d(ky_n, fu, a) \end{array} \right\} \right]$$

Although n approaches infinity,

$$d(fu, v, a) \leq \phi \left[\max \{d(fu, v, a), 0, 0, d(fu, v, a), d(fu, v, a)\} \right]$$

$$\text{i.e., } d(fu, v, a) = \phi[d(fu, v, a)] > d(fu, v, a), \text{ which yields } fu = v = u.$$

Hence, f, g, h, and k all have a single fixed point at $u = v$.

Let $w \neq u$ be an additional fixed point of f, g, h, and k to ensure uniqueness. Continuing with (1), we get

$$d(fu, gw, a) \leq \phi \left[\max \left\{ \begin{array}{l} d(hu, kw, a), d(fu, hu, a), d(gw, kw, a) \\ d(hu, gw, a), d(kw, fu, a) \end{array} \right\} \right]$$

$$= \phi \left[\max \{d(fu, gw, a), 0, 0, d(fu, gw, a), d(fu, gw, a)\} \right]$$

$$= \phi[d(fu, gw, a)] > d(fu, gw, a)$$

resulting in $w = u$ and the theorem.

This is the outcome that we obtain when we substitute $f = g$ and $h = k$.

Corollary 1. First, if f and h are self-maps on a 2 metric space (X, d) and the pairs (f, h) are sub compatible and sub sequentially continuous, then

there is a coincidence point between them.

Furthermore, if

$$d(fx, fy, a) \leq \phi \left[\max \left\{ \begin{array}{l} d(hx, hy, a), d(fx, hx, a), d(fy, hy, a) \\ d(hx, fy, a), d(hy, fx, a) \end{array} \right\} \right] \text{ for all } x, y, a \in X,$$

for every value of s from 0 to 1, the continuous function $\phi: [0,1] \rightarrow [0,1]$ is greater than s . Because of this, there is a unique shared fixed point in X for f and h .

What follows is the outcome of setting $h = k$.

Conclusion

Finally, by incorporating more generalised and flexible requirements, the framework of fixed point theory is greatly enhanced by extending compatibility principles in 2-metric spaces. Weak compatibility and its variants are examples of expanded compatibility conditions that go beyond conventional ideas of commutativity and rigorous continuity and enable the analysis of a larger class of mappings. The triadic character of the distance function adds complexity and calls for more sophisticated analytical techniques in 2-metric spaces, where this is especially crucial.

The theory's breadth and depth are further shown by using these enlarged compatibility ideas to four-map fixed point theorems. By proving that four mappings under generalised conditions have common fixed points and that these points are unique, the approach's resilience is demonstrated. In addition to bringing together many fixed point theoretic discoveries, it lays the groundwork for future generalisations that incorporate more mappings or different space structures.

In sum, by providing a more flexible and all-encompassing framework for resolving complicated issues, this study adds to the continuing advancement of nonlinear analysis. The findings have important theoretical implications and may have real-world uses in fields including computational mathematics, optimisation, and dynamical systems. These extensions should motivate future work in the subject by shedding light on the behaviour of mappings in generalised spaces and suggesting new avenues for investigation.

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