

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
NAAS Rating (2026): 4.49
Maths 2026; 11(4): 55-59
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<https://www.mathsjournal.com>
Received: 17-01-2026
Accepted: 23-02-2026

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Ring theory and pure rings and their structural applications in abstract algebra: A homological perspective

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DOI: <https://www.doi.org/10.22271/math.2026.v11.i4a.2298>

Abstract

This study examines the mathematical structure of pure rings and their fundamental role in modern abstract algebra theories. The research problem lies in determining the requirements for generalizing the concept of "emptiness" and the extent to which these properties overlap with von Neumann regular rings. The investigation adopted an analytical-deductive approach, reviewing the algebraic requirements of the pure ideal and its relationship to plane units. The research concluded that every regular ring is necessarily a pure ring, while the negation requires additional conditions related to the ring being "reduced." The significance of this achievement lies in clarifying how pure rings facilitate the study of linear systems over commutative rings, which opens new horizons for algebraic approaches in the fields of arithmetic, cryptography, and algebra in its various branches.

Keywords: Nearly pure rings, Pure rings, Purity in rings, Purity theory, Von Neumann, regular rings, Reciprocal ring

Introduction

Ring theory is considered one of the fundamental pillars upon which modern abstract algebra is built, as it represents the mathematical approach capable of describing algebraic constructs that contain both addition and multiplication operations. Since the beginning of the twentieth century, with the emergence of the achievements of prominent scientists such as Emmy Noether and Wolfgang Kroll, the scientific trend has been to investigate the structural features of closed subsets (ideals) as a pivotal tool for understanding and classifying rings.

A commutative ring R containing a product identity 1_R is called a pure ring if it has a structure such that every linear equation solvable in any extension of the ring is solvable within R itself. From a structural perspective, this concept relates to the closed subset I in the ring R , where I is said to be a pure subset if the following algebraic condition holds: for every element

$$\forall Ia \in R \exists Ib \ni Ia. Ib = aa$$

From an computational perspective, if (R) is a commutative ring containing a multiplicative identity element (1_R) , then we evaluate the concept of purity through how the ideals behave within it; an ideal (I) is defined as "pure" in the ring (R) if, for every component (a) belonging to this ideal, there exists another component (b) belonging to the same ideal that satisfies the following algebraic relationship:

$$(a * b = a)$$

This determination reflects a profound structural feature, since the presence of such a component (b) implies that the ideal possesses "local neutral elements," which makes its behavior similar to that of rings governed by the rules of perfect algebraic organization.

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Furthermore, the concept of pure rings is strongly linked to the axioms of rings; in harmonic algebra, purity is a prerequisite for ensuring the specific behavior of exact sequences, a key mechanism in contemporary algebraic studies. The equivalence between the ideal (I) being pure and the resulting ring (R/I) being a ring is one of the essential results that this research aims to demonstrate and clarify its mathematical requirements.

On the other hand, the urgent need to examine pure rings becomes apparent when approaching non-Noetherian rings, as the usual methods become inadequate to describe the internal structure of the ring. Accordingly, this research will investigate the algebraic features of these rings, highlighting the times when pure rings meet reduced rings that are stripped of nilpotent elements.

The aim of the research

The ultimate goal of this research is to present a comprehensive analytical perspective that links abstract terminology with local structural gains in group theory, as well as to identify new groups that have been rarely addressed in research, as I indicated in the abstract. This supports enriching the Arabic body of knowledge with solid content that deals with this precise mathematical field, and opens avenues for researchers to connect these concepts with geometric and algebraic theories, especially in group algebra.

Chapter One - Basic Algebraic Concepts ^[1]

In this section, we define the algebraic foundations (rules) upon which the research is based, since "purity" cannot be grasped without knowledge of the terminology of the circle and the ideals.

A comprehensive understanding of "purity" is unattainable without a rigorous familiarity with the definitions of rings and ideals.

Definition: The Ring ^[1]

Let R be a non-empty set equipped with two binary operations, addition (+) and multiplication (.). The triple (R, +, .) is defined as a "ring" provided the following axioms are satisfied:

- (R, +) is an abelian group, where the additive identity is denoted by zero (0).
- The multiplication operation (.) is associative and distributive over addition.
- The ring contains a multiplicative identity (1_R) such that $1_R \cdot a = a \cdot 1_R = a$ for every element $a \in R$.

Definition: The Ideal ^[2]

An ideal I constitutes a fundamental substructure of the ring R. A non-empty subset I is defined as an ideal if it satisfies the following:

- I, +) is a subgroup of the additive group (R, +).
- The Absorption Property: For every $r \in R$ and $a \in I$, the product $r \cdot a$ resides in I (i.e., $rI \subseteq I$).

Idempotent Elements ^[3]

The element e in the loop R is called commutative if it satisfies the condition: $(e^2 = e)$

The importance of these elements lies in the fact that they generate direct ideals, which are the building blocks of a "pure ideal."

Chapter Two - Pure Rings and Ideals ^[4]

This section addresses the core of the research, moving from comprehensive conceptions of loops to a specialized

conception of purity, which links the internal structure of the ideal and its outward behavior with the measures.

Definition of Purity in Abstract Algebra

An ideal I in a ring R is denoted as a "pure ideal" if the following algebraic property holds:

For every element a belonging to I, there exists an element b belonging to I such that

$$(a * b = a)$$

This definition computationally implies that the element (a) has an implicit multiplicative identity (Local Identity) within the ideal itself, which is formulated by the equation

$$a(1 - b) = 0.$$

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The pure ideal and its algebraic properties ^[5]

The fundamental ideal possesses a set of features that distinguish it from other familiar ideals:

1. **Idempotency:** Every fundamental ideal (I) is self-recurrent, meaning that $(I = I^2)$. This implies that the product of the elements of the ideal reproduces the ideal itself.
2. **No Nimpotent Elements:** The fundamental ideal in a reducing ring cannot contain any non-zero nimpotent elements, because the condition $a = ab$ prevents the elimination of the higher powers of the element.
3. **Interpenetration and Combination:** The interpenetration of any number of fundamental ideals in a permutation ring is also a fundamental ideal, forming a lattice structure of fundamental ideals.

Structural connection to regular von Neumann loops ^[5]

Regular von Neumann rings represent the best case of purity. A ring R is regular if for every element a there exists an element x that satisfies $(a = axa)$

- Connection: In these rings, every ideal is necessarily a fundamental ideal
- Distinction: The difference lies in the fact that "fundamental" is a broader concept; a ring may contain fundamental ideals without the ring as a whole being regular.

Characterization Theorem ^[6]

There exists a fundamental theorem stating that an ideal I is pure in group R if and only if for every ideal J in group R the following condition is met

$$(I \cap J = I * J)$$

This assignment is one of the greatest geometric and algebraic problems for testing purity, as it transforms the task of searching for an "element" into a task of "intersection and multiplication of paradigms," which facilitates the computational procedures in basic algebra.

Structural Significance of Purity ^[7]

The importance of this chapter lies in the fact that understanding "purity" enables us to break down overlapping groups into simpler parts (direct sums). The pure ideal acts as a link between groups containing identical elements and those lacking them, thus providing comprehensive insight into the classification of commutative and non-commutative groups.

Chapter Three - Purity and plane units ^[8]

This chapter connects abstract algebra and homological algebra, and it forms the basis of the research's cognitive power. The connection between the concept of purity and flatness of units is considered one of the most profound conclusions in abstract algebra and homological algebra, as this connection provides a means to examine the properties of rings through the behavior of units over them.

Definition: Flatness in Algebraic Structures ^[8]

The module M on the loop R is called "planar" if the planarity operator (Tensor Product) preserves the exact sequences of modules. In other words, if f is an anisotropic application between two modules, $(f \otimes \text{id})$ remains anisotropic.

The Fundamental Connection Theorem

Let I be an example in the commutative loop R . I is a pure example when the module formed by the division (R/I) is a plane unit on R .

Detailed Proof

We will prove the first aspect (necessity)

1. Assume that I is an intrinsic perfect in R . This means that for every a in I , there exists a b in I that satisfies $a = ab$.
2. Consider the narrow perfect sequence: $0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0$.
3. For R/I to be flat, the tensor product with any right-hand module M must preserve the variance.
4. Since every element a in I can be represented as $(a * b)$, any element in the tensor product $(I \otimes M)$ will end up zero within the sequence when the intrinsic condition is satisfied, thus ensuring that the sequence remains perfect.
5. In conclusion: Flatness here is a direct consequence of the presence of a "local equivalent" (b) , which prevents "torsion" in the structure of the resulting module.

Analysis of the Proof

1. A flat module is one in which tensor multiplication preserves the perfect sequences of modules.
2. If I is integer, then the given $(a = ab)$ ensures that any tensor multiplication with the (R/I) module will not result in skewed elements or loss of algebraic variance.
3. Flatness here embodies the flexibility of elements within the integer ideal, where no algebraic constraints prevent the linear behavior of the module.
4. Corollaries:

Based on this theorem, we conclude the following:

- a. In regular rings, all spaces resulting from division are flat spaces.
- b. Purity provides an algebraic solution for dealing with non-node rings, where flatness cannot always be achieved using conventional methods.
- c. If I is constructed with an idempotent element, then R/I is always a flat space (and even a projective one).

Chapter Four - Applied models and case studies

In this section, we will highlight the practical aspect of the purity theory by examining the behavior of some well-known ring groups and testing the extent to which the purity requirements are met in the ideal units that emerge from them.

Purity test in the correct group (Z)

The integer group Z is an example of a ring that lacks the property of purity in its non-zero perfect units.

- Let $I = n\langle i \rangle$ be a perfect unit in Z where $n > 1$. - For I to be pure, every x element in I must have a y element in I that satisfies the relation $x*y = x$.
- Let $x = n$, and look for a y that is a multiple of n such that $n*y = n$. The only solution is $y = 1$. - Since 1 does not belong to I (because $n > 1$), I is not a pure perfect unit.
- In conclusion: In the group Z , the only pure perfect units are zero and the group Z itself. This explains why measures above Z (as commutative sets) are not always flat.

Purity in the direct product group

Let's consider the group R resulting from the direct product of two fields, such as $R = Q \times Q$ (where Q is the field of rational numbers).

- Let's take the ideal unit $I = Q \times \{0\}$
- Any element in this unit is of the form $(q, 0)$
- Let's choose the corresponding element $(e = (1, 0))$. We observe that $(1, 0)$ follows the unit I . - For each element $(q, 0)$ in I , we find that: $(q, 0) * (1, 0) = (q, 0)$. - Conclusion: Since the element $(1, 0)$ follows the unit and satisfies the purity condition, I represents a pure ideal unit.
- General rule: In the direct product rings of fields, every ideal is a perfect ideal, and these rings are considered "regular".

Array loop over fields

In the $M_n(F)$ matrix ring, purity is related to the concept of "projective matrices."

- Any ideal constructed with a matrix A that satisfies $(A^2 = A)$ is a perfect ideal.
- This explanation shows that purity is not limited to commutative rings, but also includes non-commutative rings, provided there are sufficient identical elements.

Purity in abbreviated episodes

In rings devoid of zero elements i.e., $x^n = 0$ implies $x = 0$, a fundamental theorem emerges:

- "An ideal I is pure in a reducing ring R if and only if I is constructed with identical elements."
- This connection simplifies the investigation of purity; by examining the elements that satisfy $e^2 = e$, we can identify all pure ideals in the ring.

Analytical Notes on the Models

The Based on the preceding models, we observe that:

1. The absence of identical elements (as in a Z -loop) leads to a lack of purity.
2. The presence of "local neutrals" is the original measure of whether a module is level.
3. Purity serves as a criterion for the "regularity" of the loop and its proximity to the field structure.

Conclusion

- Following this comprehensive analytical investigation of the structure of "pure rings," we can deduce a set of key findings that enrich our understanding of complex algebraic structures:
- First, purity in rings represents an intermediate and flexible state that allows researchers to examine rings that do not meet Noether's conditions, thus opening avenues for dealing with infinite rings.
- Second, the research demonstrated mathematically the strong correlation between "pure idealism" and "flat

modules," showing that flatness is the other side of purity in the context of rings, facilitating tensor multiplication and precise sequences.

- Third, practical applications revealed that idempotents are the fundamental driving force behind the emergence of purity, and that von Neumann regular rings are the prime example of these properties.

Recommendations

Based on the above, the following is proposed:

1. Further study of semi-pure rings and their relationship to algebraic theory.
2. Investigation into the effect of purity in Abelian groups and its applications in modern cryptography.
3. Encouraging research into the translation of advanced algebraic concepts to enrich the Arabic library with robust specialized resources

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