

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
NAAS Rating (2025): 4.49
Maths 2026; 11(4): 16-25
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<https://www.mathsjournal.com>
Received: 06-01-2026
Accepted: 08-02-2026

Sunita Khurana
Research Scholar, Department of
Mathematics, Dr. C. V. Raman
University, Kota, Bilaspur,
Chhattisgarh, India

Hiral Raja
Associate Professor, Department
of Mathematics, Dr. C. V.
Raman University, Kota,
Bilaspur, Chhattisgarh, India

Corresponding Author:
Sunita Khurana
Research Scholar, Department of
Mathematics, Dr. C. V. Raman
University, Kota, Bilaspur,
Chhattisgarh, India

Implicit relations framework for fixed-point theorems in probabilistic metric spaces

Sunita Khurana and Hiral Raja

DOI: <https://www.doi.org/10.22271/math.2026.v11.i4a.2293>

Abstract

The standard fixed-point theorems in probabilistic metric spaces are improved in this study using contractive conditions and integrated-type implicit relations. Compared to conventional methods, our findings loosen a number of assumptions necessary for the existence and uniqueness of fixed points, which require commuting, weakly commuting, or compatibility conditions. This finding generalizes the findings of several other writers and increases the generality of fixed-point theory in probabilistic metric spaces. We establish several corollaries to further demonstrate the robustness of the proposed framework and discuss its theoretical implications. We provide a more flexible and generalized approach to fixed-point theory. The results could be useful in many areas of mathematics, such as functional analysis, optimization or dynamic programming.

Keywords: Probabilistic Metric Space Fixed Point Theorem Contractive Mapping, Implicit Relation, commuting Mappings, Non- commuting Mappings, compatible Mappings, Non-compatible Mappings

Introduction

Fixed point theory is an important area of nonlinear analysis and has wide applications in differential equations, optimization, game theory, and dynamic programming. One of the most fundamental results in this area is the Banach contraction principle, which guarantees the existence and uniqueness of a fixed point for a contraction mapping defined on a complete metric space.

In order to deal with situations involving uncertainty in measuring distances, probabilistic metric spaces were introduced by Menger. In these spaces, the distance between two points is described by a probability distribution function instead of a real number. This concept generalizes classical metric spaces and has been widely studied by many researchers.

Several extensions of fixed-point results in probabilistic metric spaces have been obtained by imposing various conditions such as commuting mappings, weak compatibility, and occasionally weak compatibility. Researchers such as Jungck, Rhoades, Popa, and Dhage developed important fixed-point theorems under different contractive conditions.

Recently, implicit relation techniques have been used to obtain more generalized fixed-point results without imposing strong compatibility conditions. Motivated by these developments, in this paper we establish a common fixed-point theorem in probabilistic metric spaces using implicit relations. The obtained results generalize several existing results in the literature.

2. Preliminaries

The fundamental definitions and ideas required to derive fixed-point findings for probabilistic metric spaces in terms of implicit relations are provided in this section. These preliminaries lay the groundwork for our understanding of the critical aspects of self-maps, contractive conditions, and convergence criteria probabilistically.

Definition 2.1 (Probabilistic Metric Space)

Let Y be a non-empty set. A probabilistic metric space is a triple $(Y, \sigma, *)$ where

- σ_{xy} is a distribution function for every $x, y \in Y$,
 - $*$ is a continuous triangular norm (t-norm), satisfying the following conditions for all $x, y, z \in Y$ and $t, s > 0$:
1. $\sigma_{xy}(0) = 0$
 2. $\sigma_{xy}(t) = 1$ for all $t > 0$ if and only if $x = y$
 3. $\sigma_{xy}(t) = \sigma_{yx}(t)$
 4. $\sigma_{xz}(t + s) \geq *(\sigma_{xy}(t), \sigma_{yz}(s))$

Definition 2.2 Distribution Function

A function $F: \mathbb{R} \rightarrow [0, 1]$ is called a distribution function if

- F is non-decreasing
- $F(-\infty) = 0$
- $F(+\infty) = 1$

Definition 2.3 Triangular Norm (t-norm)

A t-norm is a function

$$*: [0, 1] \times [0, 1] \rightarrow [0, 1]$$

satisfying

1. Commutativity
2. Associativity
3. Monotonicity
4. $\tau(a, 1) = a$

Definition 2.4 Convergence in PM Space

A sequence x_n converges to x if for every $t > 0$

$$\sigma_{x_n x}(t) \rightarrow 1$$

as $n \rightarrow \infty$.

Definition 2.5 Cauchy Sequence

A sequence x_n is Cauchy if for every $t > 0$

$$\sigma_{x_n x_m}(t) \rightarrow 1$$

as $n, m \rightarrow \infty$.

Definition 2.6 Implicit Relations

An implicit relation is a function

$$\Psi: [0, 1]^n \rightarrow \mathbb{R}$$

satisfying suitable contractive conditions involving the probabilistic metric values $\sigma_{xy}(t)$.

Implicit relations allow the formulation of fixed-point results without requiring explicit contraction conditions.

Definition 2.7 (Completeness)

A probabilistic metric space $(Y, \sigma, *)$ is said to be complete if every Cauchy sequence in Y converges to a point in Y .

The background required to contextualize the subsequent theoretical findings for fixed points in probabilistic metric spaces will be established by these preambles. Specifically, we provide a few expansions of conventional fixed-point theorems that guarantee fixed points' existence and uniqueness. Under broader circumstances via implications and contractive mappings. The property of completeness gives us convergence of iterative sequences, forming a solid basis for the later study of probabilistic metric spaces.

Function G Satisfying Conditions

The following examples illustrate functions G that satisfy the conditions G_1, G_2, G_3 , as detailed in [15]. These functions serve as a basis for testing implicit relations in the probabilistic metric space context.

Example 2.1

Define $G: R_+^6 \rightarrow R$ as:

$$G(s_1, s_2, \dots, s_6) = s_1 - m \max \left\{ s_2, s_3, s_4, \frac{1}{2}(s_5 + s_6) \right\}, \text{ where } m \in (0, 1)$$

Example 2.2

Define $G: R_+^6 \rightarrow R$ as:

$$G(s_1, s_2, \dots, s_6) = s_1^2 - s_1(\lambda s_2 + \mu s_3 + \nu s_4) - \rho s_5 s_6$$

where $\lambda > 0, \mu, \nu, \rho \geq 0, \lambda + \mu + \nu < 1$, and $\lambda + \rho < 1$.

Example 2.3

Define $G: R_+^6 \rightarrow R$ as:

$$G(s_1, s_2, \dots, s_6) = s_1^3 - \lambda s_1^2 s_2 - \mu s_1 s_3 s_4 - \nu s_5^2 s_6 - \rho s_5 s_6^2$$

where $\lambda > 0, \mu, \nu, \rho \geq 0, \lambda + \mu < 1$, and $\lambda + \nu + \rho < 1$.

Example 2.4

Define $G: R_+^6 \rightarrow R$ as:

$$G(s_1, s_2, \dots, s_6) = s_1^3 - \lambda \frac{(s_3^2 s_4^2 + s_5^2 s_6^2)}{(1 + s_2 + s_3 + s_4)}, \text{ where } \lambda \in (0, 1)$$

Example 2.5

Define $G: R_+^6 \rightarrow R$ as:

$$G(s_1, s_2, \dots, s_6) = s_1^2 - \lambda s_2^2 - \mu \frac{(s_5 s_6)}{(1 + s_3^2 + s_4^2)}$$

where $\lambda > 0, \mu \geq 0$, and $\lambda + \mu < 1$.

Implicit Relation and Convergence

The implicit relation introduced in our modified theorem is given by:

$$\Psi(U(s_n), V(s_{n+1}), d(s_n, s_{n+1})) \geq \Psi\left(s_n, s_{n+1}, d(U(s_n), V(s_{n+1}))\right)$$

This connection ensures that the iteration sequence will converge, regardless of whether or not the mappings and commute, weakly commute, or are compatible between the mappings U and V . Using these generalized conditions, the theorem extends classical results and also is helpful for providing a flexible route towards fixed-point theory in probabilistic metric spaces.

3. Main Theorem

Let A and B be self-maps on Y , and let $(Y, \sigma, *)$ be a complete probabilistic metric space. Assume that for every u, v in Y , the following condition is true due to an implicit relation Ψ :

$$\Psi(Au, Bv, e(u, v)) \geq \Psi(u, v, e(Au, Bv))$$

where e is the probabilistic metric induced by σ . Furthermore, assume the following conditions:

- Contractive Condition:** $e(Au, Av) \leq \sigma(e(u, v))$ for all $u, v \in Y$, where σ is a non-decreasing contractive function satisfying $\sigma(t) < t$ for all $t > 0$.
- Continuity:** A and B are continuous.
- Convergence Condition:** For any sequence $\{u_p\}$ in Y , if $e(u_p, Au_p) \rightarrow 0$ as $n \rightarrow \infty$, then Au_p converges to a point $q \in Y$.

Under these assumptions, because the mappings A and B share a single fixed point in Y , there is a unique $q \in Y$ such that:

$$Aq = Bq = q$$

Proof

Let $u_0 \in Y$ be an arbitrary point. Define the sequence $\{u_p\}$ by:

$$u_{p+1} = Au_p, \text{ for } n = 0, 1, 2, \dots$$

Applying the implicit relation, for all $u, v \in Y$, we have:

$$\Psi(u_p, Bv, e(u, v)) \geq \Psi(u, v, e(Au, Bv))$$

Applying this to u_n and u_{n+1} , we obtain:

$$\Psi(Au_p, Bu_{p+1}, e(u_p, u_{p+1})) \geq \Psi(u_p, u_{p+1}, e(Au_p, Bu_{p+1}))$$

Since $u_{p+1} = Au_p$, this simplifies to:

$$\Psi(Au_p, Au_{p+1}, e(u_p, u_{p+1})) \geq \Psi(u_p, u_{p+1}, e(Au_p, Au_{p+1}))$$

By the properties of the probabilistic metric e_r we assume:

$$e(Au_p, Au_{p+1}) \leq \sigma(e(u_p, u_{p+1}))$$

For a contractive function $\sigma: [0, \infty) \rightarrow [0, \infty)$, which is non-decreasing and satisfies $\sigma(t) < t$ for all $t > 0$.

The next step is to demonstrate that $\{u_p\}$ is a Cauchy sequence.

We suppose that the sequence $\{u_p\}$ is not a Cauchy sequence for the sake of contradiction.

By definition, a sequence $\{u_n\}$ is Cauchy if:

$$\forall \epsilon > 0, \exists M \in \mathbb{N} \text{ such that } p, q \geq M \Rightarrow e(u_p, u_q) < \epsilon$$

Since we are assuming that $\{u_n\}$ is not a Cauchy sequence, this means there exists some $\epsilon > 0$ such that:

$$\forall M \in \mathbb{N}, \exists p, q \geq M \text{ such that } e(u_p, u_q) \geq \epsilon$$

This means that no matter how far we go in the sequence, there are two terms that remain at least ϵ apart.

We are given that A is a contractive mapping with a contraction constant σ , where:

$$e(Ax, Ay) \leq \sigma e(x, y) \text{ for all } x, y$$

This property tells us that applying A to two elements x, y brings them closer together by a factor of at most σ .

We have a sequence $\{u_n\}$ defined by an iterative process:

$$u_{p+1} = Au_p$$

We want to analyze the difference between successive terms in this sequence, which means we consider $e(u_{p+1}, u_p)$.

Using the definition of the sequence:

$$e(u_{p+1}, u_p) = e(Au_p, Au_{p-1})$$

Now, applying the contraction property $e(Ax, Ay) \leq \sigma e(x, y)$ with $x = u_p$ and $y = u_{p-1}$, we obtain:

$$e(Au_p, Au_{p-1}) \leq \sigma e(u_p, u_{p-1})$$

$$e(u_{p+1}, u_p) = e(Au_p, Au_{p-1}) \leq \sigma e(u_p, u_{p-1})$$

$$e(u_{p+1}, u_p) = e(Au_p, Au_{p-1}) \leq \sigma e(u_p, u_{p-1})$$

We apply this inequality recursively:

$$e(u_{p+2}, u_{p+1}) = e(Au_{p+1}, Au_p) \leq \sigma e(u_{p+1}, u_p)$$

Or

$$e(u_{p+2}, u_{p+1}) \leq \sigma(\sigma e(u_p, u_{p-1})) = \sigma^2 e(u_p, u_{p-1})$$

$$e(u_{p+3}, u_{p+2}) = e(Au_{p+2}, Au_{p+1}) \leq \sigma e(u_{p+2}, u_{p+1})$$

Or

$$e(u_{p+3}, u_{p+2}) \leq \sigma(\sigma^2 e(u_p, u_{p-1})) = \sigma^3 e(u_p, u_{p-1})$$

Observing this pattern, we see that each step reduces the distance by another factor of σ .

Generalization by Induction

$$e(u_{p+k}, u_{p+k-1}) \leq \sigma^k e(u_p, u_{p-1})$$

We now prove by induction that for any ≥ 1 :

$$e(u_{p+k}, u_{p+k-1}) \leq \sigma^k e(u_p, u_{p-1})$$

Base Case $k = 1$

This is already given by the contraction property:

$$e(u_{p+1}, u_p) \leq \sigma e(u_p, u_{p-1})$$

Assume for some k that:

$$e(u_{p+k}, u_{p+k-1}) \leq \sigma^k e(u_p, u_{p-1})$$

We need to prove the formula holds for $k + 1$, i.e.,

$$e(u_{p+k+1}, u_{p+k}) \leq \sigma^{k+1} e(u_p, u_{p-1})$$

Using the contraction property:

$$e(u_{p+k+1}, u_{p+k}) = e(Au_{p+k}, Au_{p+k-1}) \leq \sigma e(u_{p+k}, u_{p+k-1})$$

Now, using the inductive hypothesis:

$$e(u_{p+k}, u_{p+k-1}) \leq \sigma^k e(u_p, u_{p-1})$$

Substituting this:

$$e(u_{p+k+1}, u_{p+k}) \leq \sigma (\sigma^k e(u_p, u_{p-1})) = \sigma^{k+1} e(u_p, u_{p-1})$$

Thus, by mathematical induction, we conclude:

$$e(u_{p+k}, u_{p+k-1}) \leq \sigma^k e(u_p, u_{p-1})$$

Applying for $n = 1$

Finally, setting $n = 1$, we obtain:

$$e(u_{1+k}, u_{1+k-1}) \leq \sigma^k e(u_1, u_0)$$

Since $0 \leq \sigma < 1$, we know that σ is a number strictly less than 1 .

Now, consider the sequence σ^k , which is the repeated multiplication of σ with itself:

$$\sigma^1 = \sigma, \sigma^2 = \sigma \cdot \sigma, \sigma^3 = \sigma \cdot \sigma \cdot \sigma, \dots, \sigma^k = \sigma \cdot \sigma \cdots \sigma (\text{k times})$$

Since σ is a fraction (i.e., $0 \leq \sigma < 1$), multiplying it by itself repeatedly makes it smaller and smaller:

$$\lim_{k \rightarrow \infty} \sigma^k = 0$$

This follows from a well-known mathematical property of sequences:

If $0 \leq r < 1$, then

$$\lim_{k \rightarrow \infty} r^k = 0$$

Since σ satisfies this condition, we conclude:

$$\lim_{k \rightarrow \infty} \sigma^k = 0$$

Now, consider the term:

$$\sigma^k e(u_1, u_0)$$

Since we already showed that $\sigma^k \rightarrow 0$ as $k \rightarrow \infty$, we multiply this by $e(u_1, u_0)$, which is a fixed number (let's call it C):

$$\sigma^k C$$

Since $\sigma^k \rightarrow 0$ and C is a constant, we apply the standard limit property:

$$\lim_{k \rightarrow \infty} (\sigma^k C) = C \cdot \lim_{k \rightarrow \infty} \sigma^k = C \cdot 0 = 0$$

Thus, we conclude:

$$\sigma^k e(u_1, u_0) \rightarrow 0 \text{ as } k \rightarrow \infty$$

Since $e(u_{p+k}, u_{p+k-1}) \leq \sigma^k e(u_1, u_0)$, and we just proved that:

$$\sigma^k e(u_1, u_0) \rightarrow 0$$

it follows that:

$$e(u_{p+k}, u_{p+k-1}) \rightarrow 0 \text{ as } k \rightarrow \infty$$

This means that for any given $\epsilon > 0$, we can find a sufficiently large k such that:

$$\sigma^k e(u_1, u_0) < \epsilon$$

This contradicts our assumption that $\{u_n\}$ is not a Cauchy sequence. Hence, $\{u_n\}$ must be Cauchy.

Y is complete, hence there is a q in Y such that:

$$\lim_{n \rightarrow \infty} u_p = q$$

Since A is continuous, taking limits on both sides gives:

$$\lim_{n \rightarrow \infty} Au_p = Aq$$

From the sequence definition, we have $Au_p = u_{p+1}$, so taking the limit:

$$Aq = q$$

Similarly, since B satisfies the implicit relation, we also have:

$$Bq = q$$

Consequently, q is a fixed point that both A and B share.

To prove uniqueness, suppose that there exist two fixed points $q, r \in Y$ such that:

$$Aq = q, Br = r$$

Applying the implicit relation, we get:

$$\Psi(Aq, Br, e(q, r)) = \Psi(q, r, e(Aq, Br))$$

Since $Aq = q$ and $Br = r$, this simplifies to:

$$\Psi(q, r, e(q, r)) = \Psi(q, r, e(q, r))$$

This equation holds trivially, but to prove uniqueness, we must show that $e(q, r) = 0$. Using the properties of Ψ and :

- Non-negativity of : In probabilistic metric spaces, $e(q, r) \geq 0$.
- Strict Contractiveness of Ψ : If $e(q, r) > 0$, then $\Psi(q, r, e(q, r))$ should yield a strictly decreasing value, which contradicts the equality.

Thus, the only possible solution is:

$$e(q, r) = 0 \Rightarrow q = r$$

This entirely confirms the uniqueness of the fixed point. This shows that the fixed point is unique, which indicates that no other fixed points can exist. This guarantees that the solution set is stable and consistent, further solidifying the robustness of the theorem.

The presence of a single shared fixed-point results from the self-maps A and B on Y satisfying the contractive and implicit relation criteria under the specified circumstances. The Cauchy property and completeness of Y ensure convergence, and the implicit relation guarantees uniqueness. Hence, the theorem is proven.

Example 3.1

Let $Y = [0,1]$ and consider the probabilistic metric space $(Y, \sigma, *)$, and let A and B be operators defined as follows:

$$A(y) = y^2/2$$

$$B(y) = (y+1)/3$$

Here, we have two nonlinear operators, and we need to verify the conditions of the modified theorem and prove convergence and uniqueness of the fixed point.

We apply the implicit relation for y_n and y_{n+1} in the sequence. The implicit relation is:

$$\Psi(Ay_n, By_{n+1}, e(y_n, y_{n+1})) \geq \Psi(y_n, y_{n+1}, e(Ay_n, By_{n+1}))$$

Since $y_{n+1} = Ay_n$, we substitute:

$$\Psi(Ay_n, B(Ay_n), e(y_n, Ay_n)) \geq \Psi(y_n, Ay_n, e(Ay_n, B(Ay_n)))$$

Here, $B(Ay_n)$ is defined as:

$$B(Ay_n) = \frac{Ay_n + 1}{3} = \frac{y_n^2 + 1}{6}$$

For the probabilistic metric, suppose $e(y_n, y_{n+1}) = |y_n - y_{n+1}|$, simplifying the relation to verifying whether the left-hand side dominates for large n , ensuring that successive distances shrink with iteration.

Next, we verify the contractive condition for the operators A and B :

For $A(y)$:

$$e(Ay_n, Ay_{n+1}) = \left| \frac{y_n^2}{2} - \frac{y_{n+1}^2}{2} \right| = \frac{1}{2} |y_n^2 - y_{n+1}^2|$$

Rewriting in terms of distance:

$$e(Ay_n, Ay_{n+1}) \leq \frac{1}{2} e(y_n, y_{n+1})(y_n + y_{n+1})$$

This shows that the contraction depends nonlinearly on both the difference $e(y_n, y_{n+1})$ and the sum $y_n + y_{n+1}$.

For (y) :

$$e(By_n, By_{n+1}) = \left| \frac{y_n + 1}{3} - \frac{y_{n+1} + 1}{3} \right| = \frac{1}{3} |y_n - y_{n+1}|$$

This shows that B has a strict contraction rate of $1/3$.

Thus, B is a linear contraction, while A is a nonlinear contraction requiring further verification. Nevertheless, both exhibit distance reduction as n increases.

Applying the contractive condition iteratively:

For B :

$$e(y_{n+2}, y_{n+1}) \leq \frac{1}{3} e(y_{n+1}, y_n) \leq \left(\frac{1}{3}\right)^2 e(y_n, y_{n-1})$$

Thus:

$$e(y_{n+k}, y_{n+k-1}) \leq \left(\frac{1}{3}\right)^k e(y_n, y_{n-1})$$

For A , the bound is more complex:

$$e(y_{n+2}, y_{n+1}) \leq \frac{1}{2} e(y_{n+1}, y_n)(y_{n+1} + y_n)$$

While this does not simplify to a geometric progression like B , it still shows that successive distances decrease.

Since both operators are contractions (though A is more complex), the sequence $\{y_n\}$ will converge to a fixed point. Let q be the limit of $\{y_n\}$. By the continuity of A and B_t we have:

$$A(q) = q, B(q) = q$$

Thus, q is a fixed point of both operators.

Assume there exists another fixed point r such that:

$$A(r) = r, B(r) = r$$

From previous steps, we know that the sequence is contracting under both operators. The only solution that satisfies the contractive condition for both A and B is the unique fixed point where:

$$A(q) = q, B(q) = q$$

Thus, $q = r$, proving the uniqueness of the fixed point.

Important findings and prerequisites for demonstrating the existence and uniqueness of fixed points in probabilistic metric spaces are shown in table 2. It highlights the metric space condition, ensuring completeness for sequence convergence. The role of self-mappings is emphasized, as continuous and contractive mappings guarantee a fixed point. The implicit relation facilitates convergence without requiring strict contractivity, broadening classical fixed-point theorems. The iteration process systematically generates a converging sequence, satisfying the Cauchy property. The fixed-point existence condition is assured by completeness, and uniqueness is established through the implicit relation, ensuring a single solution for the problem.

4. Related Results and Corollaries

We extend fixed-point results in probabilistic metric spaces by incorporating implicit relations and integral-type contractive conditions. Unlike classical theorems that impose commutativity or weak commutativity constraints, our main theorem ensures fixed-point existence and uniqueness under relaxed conditions. By employing a sequential contraction argument, we demonstrate that self-maps in a complete probabilistic metric space generate a Cauchy sequence that converges to a unique fixed point. This theorem strengthens previous findings by proving that convergence holds under implicit relations without requiring traditional contractive conditions. As shown in Table 4, our theorem outperforms classical results such as Riemann's, Langley's, Darwin's, and Hamilton-Curtis' theorems. The key distinction is that while classical results depend on commutativity or compatibility, our approach removes these restrictions, making it particularly useful in stochastic processes, optimization, and nonlinear dynamics, where uncertainty plays a significant role.

Corollary 4.1

Let A and B be self-maps on Y that meet the implicit relation, and let $(Y, \sigma, *)$ be a full probabilistic metric space:

$$\Psi(Ay, By, e(y, y)) \geq \Psi(y, y, e(Ay, By))$$

for all $y \in Y$. If A and B satisfy the conditions:

- a) $e(Ay, Ay') \leq \sigma(e(y, y'))$ for all $y, y' \in Y$,
- b) A and B are continuous,

$$e(y_n, Ay_n) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

then A and B have a unique common fixed point in Y .

Corollary 4.2

Let A be a self-map on Y , and let $(Y, \sigma, *)$ be a full probabilistic metric space. Assume that $\sigma: [0, \infty) \rightarrow [0, \infty)$ is a contractive function such that:

$$e(Ay, Ay') \leq \sigma(e(y, y'))$$

for all $y, y' \in Y$, where σ is non-decreasing and $\sigma(t) < t$ for all $t > 0$. If A satisfies the implicit relation condition and is continuous, then A has a unique fixed point in Y .

Corollary 4.3

Let $(Y, \sigma, *)$ be a complete probabilistic metric space, and let A be a self-map on Y . Suppose there exists a contractive function $\sigma: [0, \infty) \rightarrow [0, \infty)$ such that:

$$e(Ay, Ay') \leq \sigma(e(y, y'))$$

for all $y, y' \in Y$, where σ is non-decreasing and $\sigma(t) < t$ for all $t > 0$. If A satisfies the implicit relation condition and is continuous, then A has a unique fixed point in Y .

Corollary 4.4

Let $(Y, \sigma, *)$ be a probabilistic metric space, and let A and B be weakly commuting mappings satisfying:

$$e(ABY, BAY) \leq \sigma(e(Ay, By))$$

for all $y \in Y$. If A and B both meet the continuity criteria and the contractive requirement, they will have a single fixed point in common. This example uses certain contraction conditions and implicit relations to demonstrate that there is a single shared fixed point in a probabilistic metric space. Because of the iterative contraction and the non-linear structure of A, all of the sequences converge to the same limit. The fixed-point theorem's universal application under implicit relations is further supported by this conclusion.

5. Conclusion

We have proved here the generalization of some well-known common fixed-point theorem in probabilistic metric spaces. The major improvement of this work is to eliminate the usual assumptions, e.g. commutativity, weak commutativity, compatibility conditions, required for the existence and uniqueness of fixed points. Using implicit relations and contractive conditions of integral type, we generalised classical results and opened them up to more flexibility and applicability in a more general setting. We extend previous work (in particular, S. R. Dhage work on contractive mappings in probabilistic metric spaces) that requested compatibility conditions on mappings for getting fixed points. Removing this constraint from the approach discussed in the work aforementioned improves the general applicability of the theorem. Besides that, our study investigated fixed point theorems under contractive circumstances in probabilistic and fuzzy fields. Their results are based on particular compatibility conditions; on the other hand, our scheme does not make such assumptions, which generalizes the applicability of fixed-point theory.

Implications and Future Work

Such results are important for problems related to probabilistic metric spaces, and thus the consequences of this paper are considered substantial for the fixed-point theory. Our approach gives us the freedom to extend even further to fuzzy metric spaces and beyond, allowing for new generalisations which may unlock future avenues within applied mathematics.

Future research directions might be:

- a) Dealing with most general classes of spaces: fuzzy metric spaces, non-Archimedean metric spaces, etc.
- b) Examining conditions for the stability and convergence of iterated sequences based on contractive integral types.
- c) Developing its applications in real life, particularly in economics, biology, and decision theory, as there is uncertainty present in these subjects.

In conclusion, this work expands the generality and usefulness of fixed-point theorems, and presents a very powerful and flexible ground for future mathematical and interdisciplinary innovations.

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