

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
NAAS Rating (2026): 4.49
Maths 2026; 11(3): 107-110
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<https://www.mathsjournal.com>
Received: 30-11-2025
Accepted: 15-01-2026

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A comprehensive study on the evolution and applications of fixed-point theory in modern mathematical analysis

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DOI: <https://www.doi.org/10.22271/math.2026.v11.i3b.2283>

Abstract

Fixed-point Theory provides a common ground for the solution of equations and optimization problems in many different contexts. The theoretical underpinnings, development, and many uses of fixed-point theory are investigated in this article. The subject starts with the proofs of fixed-point theorems in metric and normed spaces and follows their extension to topological and partially ordered structures, starting with Poincaré, Brouwer, and Banach. Nonlinear analysis, computational mathematics, dynamic systems, and game theory all rely on the idea of a fixed-point, which represents an invariant under a particular function. Solving functional, differential, and integral equations, as well as demonstrating their uniqueness, are the primary areas of theoretical focus. The significance and flexibility of the theory in both pure and applied mathematics are demonstrated in the recent extensions of fixed-point results to fuzzy, probabilistic, and complex-valued metric spaces, which are highlighted in the paper's conclusion.

Keywords: Fixed-point theory, nonlinear analysis, metric spaces, mathematical modelling, functional equations, topological structures

Introduction

Not only do fixed-point methods have theoretical uses, but they are also heavily utilized in computational mathematics. Iterative methods for solving nonlinear and linear equations share the property of convergent to a fixed-point. The Newton-Raphson method is a gold standard in numerical analysis; it is an iterative strategy to finding fixed-points of a function that are related to the original function's derivative. Fixed-points play a crucial role in economic theory, particularly when investigating equilibrium points. Every player's strategy is perfect in light of their opponents' moves when the game reaches a Nash equilibrium. The best-response correspondence is the name given to this idea in game theory. A famous example of a fixed-point theorem in action proved that games with a finite number of players do in fact contain such equilibria. General equilibrium theory, which examines the economy in its entirety, uses fixed-points to model and prove the existence of a state of equilibrium where all markets are clear.

The use of fixed-point theory can be extended to many more scientific domains. As with stationary points in a growth function, fixed-points in a biological model of population dynamics represent equilibrium populations. Ecologists characterize ecosystems where the number of species remains constant across time using the idea of fixed-points. In physics, the same is true: stable states of physical systems, such as particle equilibrium configurations in a potential field, are characterized by fixed-points.

Review of related studies

Devi, Konthoujam. (2024). Where does stability and equilibrium, represented as a function regressing to itself, lie? An essential branch of mathematics, this is the domain of fixed-point theorists. Focusing on metric spaces, Banach spaces, and Hilbert spaces, this study investigates theorems of fixed-points with the goal of demonstrating their usefulness and importance. In full metric spaces, the Banach Fixed-point Theorem ensures that there exist unique fixed-points

for contraction mappings, and Brouwer's Fixed-point Theorem states that any continuous function mapping a compact convex set to it will have a fixed-point. There has been an expansion of these classical results to include generalized distance functions, new types of contraction mappings, and enhanced relevance to optimization issues, control theory, and dynamic systems. Additionally, the paper demonstrates the significance of the Schauder Fixed-point Theorem in Banach spaces for nonlinear operator analysis and provides it. Studied in relation to optimization techniques and nonlinear integral equations, fixed-point findings in Hilbert spaces are emphasized for their applicability to variational methods and engineering. Some examples of new trends include combining computer approaches with classic fixed-point methods and studying fixed-point results in probabilistic and fuzzy situations. This article traces the historical development of fixed-point theory, connecting theoretical mathematics with practical applications. Finally, it provides directions for future research on fixed-point theory and its modern mathematical applications.

Shalini,. (2024). When it comes to mathematical modeling and optimization, fixed-point theory provides crucial tools for understanding and solving complicated issues, making it an integral part of many mathematical disciplines. This paper surveys the uses of fixed-point theorems, like Banach and Brouwer's, in many fields, including economics, engineering, biology, and more. This presentation examines case studies and practical applications to highlight the adaptability of fixed-point theory as a powerful tool for tackling current difficulties in mathematical modeling and optimization. In order to improve theoretical knowledge and solution techniques, the results highlight the importance of incorporating fixed-point principles into various analytical frameworks.

Bucur, Amelia. (2017) ^[5]. Numerous theoretical and applied domains rely on fixed-point theory. These include nonlinear analysis, dynamic systems theory, mathematics of fractals, mathematical economics (game theory, equilibrium problems, and optimization problems), mathematical modeling, and variational and linear inequalities. A few standards for the applications of fixed-point theory are given in this study. Additionally, this study discusses the possibility of applying the findings of the fixed-point theory to mathematical quality modeling.

Historical roots of fixed-point theory

Ancient mathematicians such as Ptolemy and Archimedes laid the groundwork for fixed-point theory via their investigations of geometry and mechanics, which in turn led them to indirectly address fixed-point issues. The idea of a fixed-point, or the final endpoint of a function, did not explicitly develop until much later.

René Descartes's work from the 17th century was a major milestone in the evolution of Fixed-point Theory. Fixed-point Theory relies heavily on Descartes's work in analytical geometry, which allowed us to comprehend the behavior of functions and their graphs. An early version of fixed-point analysis could have included solving equations by locating the point where a function intersects the identity line.

However, Fixed-point Theory did not develop into its own subfield of mathematics until the nineteenth century. Augustin-Louis Cauchy laid the groundwork for the study of fixed-points with his concepts of limits and continuity in analysis. The foundation for later, more formal definitions and theorems was laid by Cauchy's work.

The study of polynomial equations was greatly advanced by Joseph-Louis Lagrange and Carl Friedrich Gauss. We looked at what it takes for a polynomial to have roots, which are the places at which the related function is fixed. Their findings reveal a more fundamental concept and raise the possibility that fixed-points are useful in contexts other than polynomial equations.

After Charles Émile Picard's work in the late 1800s, Fixed-point Theory started to be formally established. In Picard's theorem, an older version of the fixed-point theorem, it is guaranteed that certain differential equations have solutions. His research proved that fixed-points are fundamental to mathematical analysis and have important consequences, rather than being an unusual anomaly.

Luitzen Egbertus Jan Brouwer was a significant player in the development of topological approaches in early 20th-century Fixed-point Theory. According to the well-known Brouwer's Fixed-point Theorem, any continuous function in Euclidean space with a closed, bounded, and convex set as its boundary must have a fixed-point. It was a huge step forward in theory, giving mathematicians a new tool and having an effect on areas as diverse as economics and game theory. New research directions were made possible once Brouwer transformed Fixed-point Theory from an analytical to a topological idea.

The Banach Fixed-point Theorem, often called the Contraction Mapping Theorem, was developed by Stefan Banach in 1922, after Brouwer's pioneering work. The existence of separate fixed-points in contraction mappings on fully metric spaces is assured by Banach's theorem. Numerous fields have found applications for this finding, such as numerical analysis, differential equations, and dynamic systems.

In the mid-twentieth century, Fixed-point Theory saw an expansion into higher dimensions and abstraction. The field was substantially advanced by the contributions of mathematicians like Lefschetz, who established the Lefschetz Fixed-point theory, and Shizuo Kakutani, who extended Brouwer's theory to set-valued functions. The theorem of Lefschetz links fixed-points with topological invariants and is a strong instrument in algebraic topology for continuous mappings on compact oriented manifolds.

In the second part of the twentieth century, game theorist John Nash and others substantially contributed to the development of Fixed-point Theory. Our work on equilibrium points in non-cooperative games, which directly applies fixed-point theorems, was the subject of Nash's Nobel Prize in Economics. His work proved that Fixed-point Theory may be used in areas other than mathematics, such as biology, economics, and social science.

Modern computer technology has allowed for new insights into Fixed-point Theory since its inception in the late 20th and early 21st centuries. Optimization, algorithmic game theory, and dynamical systems are just a few of the many fields that have advanced greatly with the advent of computing and numerical techniques that make it possible to effectively calculate fixed-points in complex systems. Because of its possible applications in artificial intelligence, data analysis, and machine learning, research into the dynamic field where Fixed-point Theory meets computational mathematics is flourishing.

Over the years, Fixed-point Theory has shown time and time again that it can bring together mathematical disciplines that at first glance seem to have nothing in common. Its ideas are applicable to numerous areas, including manifold topology, analysis of differential equations, and polynomial geometry. The theory's evolution parallels that of mathematics as a whole,

which is moving toward more theoretically complicated problems.

Fixed-point

An element or point inside the scope of the purpose that the mapping associates with itself is called the fixed-point of the mapping. The fixed set is the aggregate of all fixed-points.

1. If S is a non-empty set and H is a self-map from S to S , then an element s in S is a fixed-point of H if and only if $H(s) = s$.
2. A point is considered fixed in a mapping H from set S to set of non-void subsets of S if and only if $\{s \in FH = \{s \in S \mid Hs = s\}\}$.
3. For 'p' to be a fixed-point of H , the iterated purpose order $H(H(p)...) = n$ $n(p)$ must be equal to p .

Since $H(2) = 2$ and $H(s) = s^2 - 3s + 4$, we can deduce that 2 is a fixed-point of H .

However, in the case when S is a non-empty set and ϕ is a self-map of S , any point s in S is deemed to be a fixed-point of S when $\phi(s)$ is identical to s . A fixed-point of the mapping ϕ from set S to the set of non-empty subsets of S is any element s in set S that is also a member of $\phi(S)$, assuming that ϕ is a functional of S .

The set $a\phi = \{s \in S \mid \phi s = s\} \neq \emptyset$ is defined as the "fixed set" of ϕ .

Observation One: Not every function has a fixed-point. The function $\phi: \mathbb{R} \rightarrow \mathbb{R}$ does not have a fixed-point since there is no real number s in $\mathbb{R}$ that can be equal to $s + 1$. It transfers from the set of real numbers $\mathbb{R}$ to itself and has the property that $\phi(s) = s + 1$. It is common practice to utilize a fixed-point "s" when graphically displaying the point $(s, H(s))$ that lies on the line $y = s$.

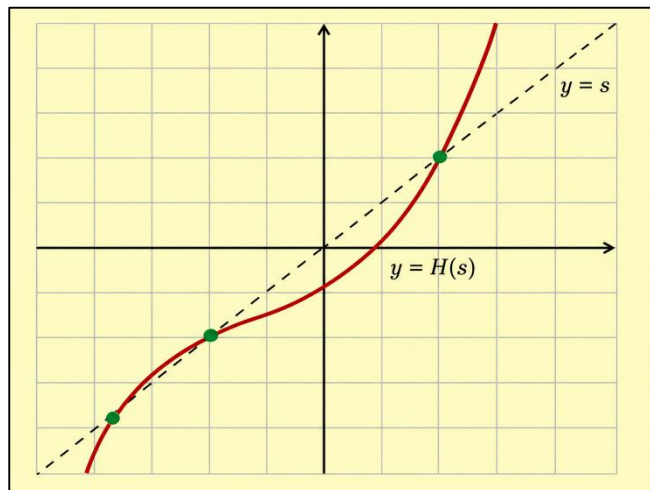


Fig 1: Function Characterized by Three Fixed-points

Attractive Fixed-point

Perfect Spot for Nothing. For any value of s in the domain that is sufficiently near to s_0 , this iterative function sequence converges to s_0 . This is referred to as an attractive fixed-point of H . Starting with 1, the natural cosine function approaches its maximum value of 0.739085133 (rounded to nine decimal places). To rephrase, the equation $s_{n+1} = \cos s_n \rightarrow 0.739085133$ is the result when the cosine graph connects with $y = s$.

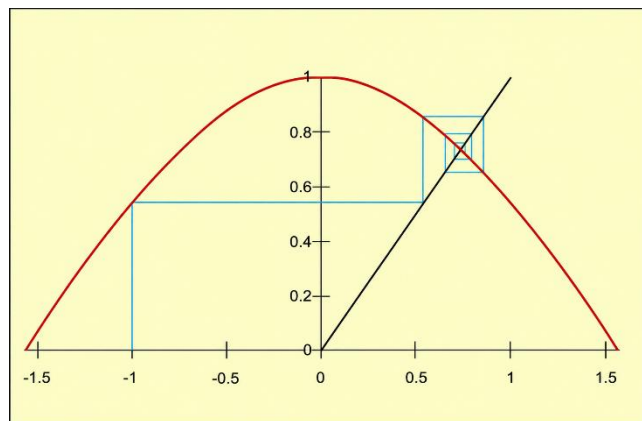


Fig 2: The Fixed-point Iteration $s_{n+1} = \cos s_n$ with initial value $s_1 = 1$

The solutions to the equation $H(s) = 2s$, for instance, have a stable point at $s = 0$, but they diverge fast for all values other than zero.

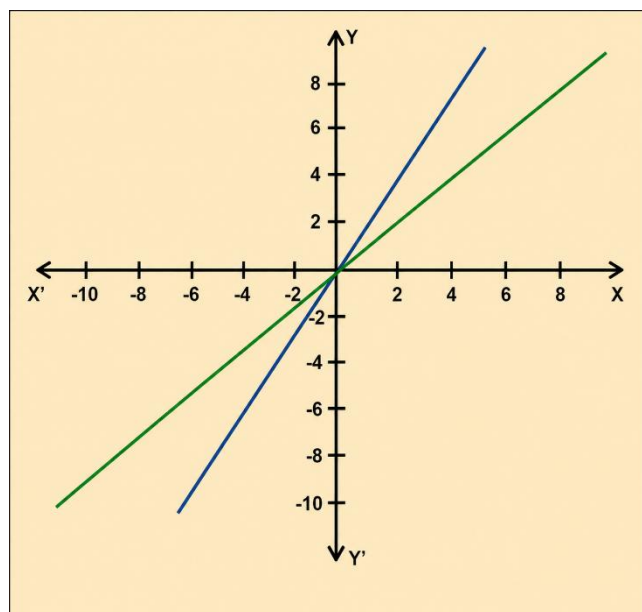


Fig 3: Divergence of $H(s) = 2s$

Applicability of fixed-point theory

An essential part of mathematical analysis is solving differential equations using Fixed-point Theory. As a representation of a situation or process that changes over time, differential equations are often used in the fields of physics, engineering, and economics. A steady-state or equilibrium solution to these equations is often indicated by the existence of a fixed-point. Given that it relies on a fixed-point theorem applied to an integral operator, the Picard-Lindelöf theorem ensures that certain ordinary differential equations have distinct and actual solutions. In this setting, fixed-points are an excellent tool for qualitative exploration of dynamical systems as they reflect solutions that satisfy the original differential equation.

The Contraction Mapping Theorem, often known as Banach's Fixed-point Theorem, is a significant finding in functional analysis. Contraction mappings, in which the distance between points is continually reduced, have unique fixed-points in entire metric spaces. This theorem gives a way to prove it. Significant theoretical ramifications follow from this, particularly with respect to solutions of linear and nonlinear

equations. Numerical analysis makes use of iterative procedures based on contraction mapping concepts; for instance, the Newton- Raphson method and others for finding the roots of real-valued functions. An essential theorem in computational mathematics, Banach's theorem guarantees convergence to a solitary fixed-point, lending credence to various numerical approaches.

The solution of integral equations, which use functions rather than integers, is another domain where Fixed-point Theory finds application. Integral equations are widely used in many disciplines for the purpose of modeling heat conduction, fluid dynamics, and population dynamics, among many others in engineering, biology, and physics. By recasting as fixed-point problems, the solution function of several integral equations may be transformed into a fixed-point of an associated integral operator. In addition to uniqueness and existence findings, this approach offers iterative methods to estimate solutions, which are useful for real-world computations.

The optimization capabilities of Fixed-point Theory are shown by its examination of convex functions and variational inequalities. Locating a fixed-point on the gradient or subgradient of a function is often seen as finding its minimum or maximum in optimization issues. The famous Brouwer Fixed-point Theorem states that, on compact convex sets, there are fixed-points for continuous functions, and this situation is very relevant to that conclusion. Machine learning, signal processing, and operations research rely on convex optimization methods like gradient descent, which it theoretically establishes. If everything goes according to plan, the fixed-point approach will make these algorithms' analysis and convergence much easier.

Research into dynamical systems, which examines how systems behave as they evolve over time, also heavily relies on Fixed-point Theory. A system is said to have achieved a fixed-point when it maintains a constant value across time. The stability of these equilibrium points is an important issue in dynamical system analysis because they dictate the long-term repercussions of modest perturbations. These stability features may be investigated using Fixed-point Theorems. By analyzing the stability of iterated functions in discrete dynamical systems, one may find long-term behavior shown by eigenvalues and the Jacobian matrix.

A key premise in Nash equilibrium makes Fixed-point Theory a potent tool in game theory. In this state of balance, all player's moves are perfect complements to one another. Based on Kakutani's Fixed-point theory an extension of Brouwer's theory to set-valued functions Nash deduced that finite games have an equilibrium point. In fields like politics, economics, and evolutionary biology, where models and assessments of strategic interactions are common, this finding has far-reaching consequences. A lot of social science researchers rely on Fixed-point Theory because it can guarantee that equilibrium solutions really exist in complicated, multi-agent systems.

In addition to these more traditional uses, Fixed-point Theory has been revitalized in contemporary mathematical analysis via the links between geometry, algebra, and topology. A consequence that establishes a connection between the quantity of fixed-points in a continuous map and the topological invariants of the space supporting it is the Lefschetz Fixed-point Theorem. In algebraic topology, this theorem has important consequences for investigating the characteristics of manifolds and other geometric objects. A more general application of Fixed-point Theory is the study of structures and symmetries in algebraic geometry and categories via the fixed-points of functors and endomorphisms.

Fixed-point Theory has elevated to a much higher echelon in mathematical analysis with the development of computational resources. The capacity to efficiently calculate fixed-points has grown into a major obstacle in domains like algorithmic game theory, optimization, and numerical analysis. Homotopy techniques and continuation methods based on Fixed-point Theorems are two examples of algorithms that have been useful in solving difficult, high-dimensional problems in many scientific and engineering fields. In the case of large-scale systems, when conventional analytical techniques may not be applicable, computational methods such as these yield superior findings.

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