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An analysis of discriminants and integral bases in pure number fields

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Abstract

The study of pure number fields plays a central role in algebraic number theory due to their simple defining equations and deep arithmetic properties. In this paper, we present a detailed analysis of the discriminants and integral bases of selected pure number fields. Within this framework, the paper develops an explicit and computable expression for the discriminant of the field (K) . The formula depends exclusively on the prime power factorizations of (a) and (n) , thereby avoiding unnecessary complexity and allowing for direct computation. This approach clarifies the role of ramification in pure number field extensions and provides insight into the structure of the corresponding rings of integers. The results obtained not only simplify discriminant calculations but also contribute to a deeper understanding of integral bases in pure number fields, offering a useful foundation for further theoretical and computational research.

Keywords: Pure number fields, algebraic, discriminant formula, integral basis, field extensions, prime ramification

1. Introduction

In the study of algebraic number fields and their arithmetic structure, the study of integral bases and discriminants is important to algebraic number theory. When an integer's n -th root is joined to the rational numbers in a specific way (usually written as $K = (\sqrt[n]{m})$), where m is a nonzero integer that is not a n -th power, the resulting number field is known as a pure number field. Deep algebraic concepts like rings of integers, field extensions, and factorization features can be better understood with the help of these fields, which offer a rich but tractable class of instances. A way to understand how the arithmetic features of the base integer m impact the field's structure is to examine discriminants and integral bases in these types of fields.

All algebraic integers contained in a number field have a "coordinate system" that is provided by an integral basis of that field. More precisely, it is a free abelian group that forms a basis of the ring of integers O_K . If an integral basis can be found, then every algebraic integer in the field may be uniquely written as a linear combination of basic elements with integer coefficients, which is why finding one is so important. Delicately depending on the values of m and n , the nontrivial task of obtaining an integral basis in pure number fields is routinely performed. Although integral bases are known to work in certain elementary situations, such quadratic fields, higher-degree pure fields (like cubic, quartic, or nonic fields) pose formidable theoretical and computational obstacles.

A critical invariant conveying important arithmetic information, the discriminant of a number field is closely related to the concept of an integral basis. Under field embeddings, the selected basis interacts as shown by the discriminant, which is the determinant of the matrix produced by the traces of products of basic elements. The discriminant, when applied to pure number fields, reveals information about prime ramification, algebraic integer density, and associated lattice architecture in Euclidean space. In addition, it is crucial for comprehending the behavior of prime numbers in O_K , whether they divide or stay unchanged in the extension field.

Discriminants and integral bases interact in a very important way. When calculated with regard to an integral basis, the discriminant's absolute value is sign-invariant, but its value is basis-dependent. For this reason, it is an effective method for checking if a specific collection of items constitutes an integral foundation. Researchers in the area of pure numbers frequently seek out clear formulae or criteria that link the integral basis structure to the matching discriminant and the parameter u . Simplifying computations and classifying number fields with comparable attributes rely on these linkages.

Researching discriminants and integral bases is important for more than just academic purposes. Computational number theory, algebraic geometry, and cryptography are some areas that could benefit from these ideas. Modern mathematical software relies on rapid computing of integral bases for algorithmic implementations, while the study of Diophantine equations and field ramification makes use of discriminants. Such studies also add to larger research issues in pure number fields, such as unique factorization and class number determination.

Finally, algebraic number fields' arithmetic and structural features can be better understood by studying discriminants and integral bases in pure number fields. In algebraic number theory, mathematicians can gain a better understanding of the theory and practical computations by studying the behavior of these invariants in relation to the defining integer m and the degree of the extension. Research in this area is still going strong, with people trying to find new connections inside these intriguing algebraic structures, improve computational tools, and create broader methodologies.

2. Review of previous studies

Rani, Pooja & Sharma, Alok. (2026) ^[15]. The discriminant and integral basis are fundamental concepts that illuminate the structure and mathematical properties of pure number fields when their study is undertaken. The discriminant, which stores information about the field's ramification and the geometry of its ring of integers, affects the algebraic invariants and the arithmetic behavior of a number field. An integral basis, which is a collection of elements in the ring of integers that constitute a basis over the integers, allows one to study the characteristics of the field, such as the ideal class group structure and the solutions to Diophantine equations. These principles have numerous important applications in number theory, such as class number analysis, Hilbert symbol computing, and ideal class group determination. Finding minimal integral bases for fields with complicated ramification and computerizing discriminants for fields of high degree are both difficult jobs. Despite the great strides we have taken, we still need a better grasp of algebraic number theory and more sophisticated computational methods to tackle these problems. The study of integral bases and discriminants is theoretically and pragmatically vital in mathematical physics, coding theory, and cryptography because the algebraic features of number fields impact the efficiency and safety of different algorithms.

Kchit, Omar. (2024) ^[14]. The pure number field $K = \mathbb{Q}(\alpha)$ is defined as the set of all real numbers that can be simplified to a monic irreducible polynomial $x^p - a \in \mathbb{Z}[x]$, where p is a rational prime and r is a positive integer. Assume that K is an integer ring and call it ZK . An integral basis of ZK is computed and the monogeneity of K is studied in some specific circumstances in this article. Jakhar, Anuj & Sangwan, Neeraj. (2022) ^[1]. The algebraic number field $K = \mathbb{Q}(\theta)$ is defined as θ satisfying an irreducible polynomial $x^p - a$ over the field $\mathbb{Q}$ of rationals, and the ring of algebraic integers of K is denoted by ZK . We give the precise power of every prime that divides the index of the subgroup $Z[\theta]$ in ZK in this article. We also provide a foundation for K that is p -integral for every prime p . An integral basis of K can be constructed using these p -integral bases, as shown by the examples.

Jakhar, Anuj *et al.*, (2020) ^[7]. For each prime p dividing n , either pa or the largest power of p dividing a is coprime to p . This condition is obviously satisfied when a and n are coprime or when a is square free. Suppose $K = \mathbb{Q}(\sqrt[n]{n \& a})$ is an extension of degree n of the field $\mathbb{Q}$ of rational numbers. The discriminant of K , which is defined explicitly in terms of the prime powers that divide a , n , is presented in the article.

Jakhar, Anuj & Sangwan, Neeraj. (2018) ^[2]. The field $\mathbb{Q}$ of rational numbers has an extension $K = \mathbb{Q}(\theta)$, which is defined as the set of all polynomials $x^p - a$ of prime degree that belong to $\mathbb{Z}[x]$. An integral basis for K is provided explicitly in this study by utilizing solely elementary algebraic number theory. While it is known that such fields have an integral basis, we provide a slightly different and simpler definition of the integral basis. A brief demonstration of the discriminant formula for these types of fields is also provided.

Gaál, István & Remete, László. (2016) ^[9]. Consider the integer m , which is not divisible by 2. We demonstrate that the integral bases of the fields have a periodic structure in terms of m . The length of the era is displayed. We state the integral bases directly and compute the index forms of K explicitly. This allows us to describe the homogeneity of these fields in many instances. By taking the index forms in their explicit form, a novel technique is developed that allows us to obtain new results on monogeneity and, as a byproduct, numerous previous results. Because our description of the homogeneity of pure fields is nearly exhaustive.

Discriminants and integral bases in extensions defined by $x^n - a$

Our objective in this study is to provide a formula for the discriminant of n -th degree fields of the form $\mathbb{Q}(\sqrt[n]{a})$ in terms of prime powers dividing a and n , where for every prime p dividing n , either p does not divide a or the highest power of p dividing a (to be represented by $v_p(a)$) is coprime with p . Our demonstration here is based on the basic Theorem of Ore concerning Newton polygons and is more or less self-contained, while Gassert in uses a method provided by Montes and refined in to get a formula for the discriminant of $\mathbb{Q}(\sqrt[n]{a})$ using this hypothesis.

Put another way, we demonstrate:

Theorem 1. Here, θ is a root of an irreducible polynomial $f(x) = x^n - a$ belonging to $\mathbb{Z}[x]$, and K is an algebraic number field with discriminant d_K . The prime factorizations of n and $|a|$ are $\prod_{i=1}^k p_i^{s_i}$, $\prod_{j=1}^l q_j^{t_j}$ respectively. The integers $\gcd(n, t_j)$, $\frac{n}{p_i^{r_i}}$ and $v_{p_i}(a^{p_i-1} - 1) - 1$ represent m_j , n_i , and r_i , respectively. Take it for granted that $v_{p_i}(a)$ is either equal to zero or coprime to p_i for every integer i . After that

$$d_K = (-1)^{\frac{(n-1)(n-2)}{2}} \operatorname{sgn}(a^{n-1}) \left(\prod_{i=1}^k p_i^{v_i} \right) \prod_{j=1}^l q_j^{n-m_j},$$

the value of v_i is either $ns_i - 2n_i$ or ns_i , depending on whether r_i is greater than 0 or not, where $\sum_{j=1}^{\min\{r_i, s_i\}} p_i^{s_i-j}$.

The next result follows naturally from the previous theorem.

Corollary 1. Here, p is a prime number and $a = \pm 1$ is an integer that is not divisible by squares. K is equal to $\mathbb{Q}(\theta)$, where θ is a root of $x^{p^s} - a$. The value of d_K is $v_p(a^{p-1} - 1) - 1$, where v is equal to $\sum_{j=1}^{\min\{r_i, s_i\}} p^{s-j}$ or sp^s , depending on whether $r > 0$ or not.

For the unique situation where $p = 2$ and $s = 3$, the resultant formula is readily apparent. This corollary of the previous one, for $K = \mathbb{Q}(\theta)$, can be repeated in the same way as in:

$$d_K = \begin{cases} -2^{24}a^7, & \text{if } a \equiv 2, 3 \pmod{4}, \\ -2^{16}a^7, & \text{if } a \equiv 5, 13 \pmod{16}, \\ -2^{12}a^7, & \text{if } a \equiv 9 \pmod{16}, \\ -2^{10}a^7, & \text{if } a \equiv 1 \pmod{16}. \end{cases}$$

Rapid deduction from Theorem 1 leads to the following consequence, which is partially proven in.

Corollary 2. Consider K as an extension of $\mathbb{Q}$ where θ is an irreducible polynomial $x^n - a$ over $\mathbb{Z}$, and let $\mathbb{Q}(\theta)$ denote this extension. If a is not square free and for any prime p that divides n , $p^2 \nmid (a^{p-1} - 1)$, then the set $\{1, \theta, \dots, \theta^{n-1}\}$ is an integral basis of K .

3. Computation of discriminant via index of algebraic integers and newton polygons

Based on a fundamental finding, we may say that if $n = \prod_{i=1}^k p_i^{s_i}$, $|a| = \prod_{j=1}^l q_j^{t_j}$, $f(x) = x^n - a$, θ , d_K are as in Theorem 1, A_K is the ring of algebraic integers of K , and $N_{K/\mathbb{Q}}$ is the norm map, then θ is equivalent to θ .

$$d_K[A_K : \mathbb{Z}[\theta]]^2 = (-1)^{\binom{n}{2}} N_{K/\mathbb{Q}}(f'(\theta)) = (-1)^{\binom{n}{2}} N_{K/\mathbb{Q}}(n\theta^{n-1}) = (-1)^{\frac{(n-1)(n-2)}{2}} n^n a^{n-1}.$$

Once the precise powers of each p_i and q_j that divide $[A_K : \mathbb{Z}[\theta]]$ are known, the value of d_K may be calculated. Since they are simpler, we begin with the primes q_j dividing a . Primes that divide n but do not divide a can be generated in two steps as $v_{p_i}([A_K : \mathbb{Z}[\theta]])$. In the first part, we address the case where n is a prime number. We next create a relationship between $v_{p_i}([A_K : \mathbb{Z}[\theta]])$ and $v_{p_i}([A_{K_i} : \mathbb{Z}[\theta_i]])$, where A_{K_i} is the set of algebraic integers of $K_i = \mathbb{Q}(\theta_i)$, where $\theta_i = \theta^{n_i}$ and $x^{p_i^{s_i}} - a$ is a polynomial. We utilize a specific instance of the Theorem of Index of Ore to determine the prime powers that divide these indices; to do so, we require the concept of Newton polygon, which is discussed below.

" $\mathbb{Z}_p$ " means the ring of p -adic integers and " $\mathbb{F}_p$ " means the field with p elements throughout. The p -adic valuation of c in $\mathbb{Z}_p$ is denoted by $v_p(p) = 1$, and the image of c under the canonical homomorphism from $\mathbb{Z}_p$ onto $\mathbb{F}_p$ is denoted by $\bar{c}$.

Definition: Consider the polynomial $g(x) = \sum_{j=0}^n c_j x^j$ over $\mathbb{Z}_p$, where $c_0 c_n \neq 0$, and assume that p is a prime number. For every non-zero term $c_i x^i$, we create the set by associating a point $(n-i, v_p(c_i))$.

$$P = \{(j, v_p(c_{n-j})) | 0 \leq j \leq n, c_{n-j} \neq 0\}.$$

The polygonal route created by the lower edges along the convex hull of points of P is known as the p -Newton polygon of $g(x)$ or the Newton polygon of $g(x)$ with regard to p . Keep in mind that while you work your way from left to right, the edge slopes get steeper.

Definition: Assume that $g(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0$ is a polynomial over $\mathbb{Z}_p$. Then, the p -Newton polygon of $g(x)$ has one edge with a positive slope λ , meaning that the minimum value of $v_p(a_{n-1})$ for all integers from 1 to n equals $v_p(a_0)n$, which is equal to λ . Assume that e is the smallest positive integer such that $e\lambda$ is a member of $\mathbb{Z}$. A polynomial $T(Y) \in \mathbb{F}_p[Y]$ that is not divisible by Y and has a degree of $\frac{n}{e} = t$ (say) is associated with $g(x)$.

$$T(Y) = Y^t + \sum_{j=1}^t \overline{\left(\frac{a_{n-ej}}{p^{ej\lambda}} \right)} Y^{t-j}.$$

Specifically, we shall refer to $T(Y)$ as the polynomial that is associated with $g(x)$ with regard to p .

Example. Prove that $g(x) = (x+5)^4 - 5$. It is easy to verify that there is only one edge with a slope of $\kappa = 1/2$ in the 2-Newton polygon of $g(x)$. $T(Y) = Y^2 + Y + \bar{1}$ belongs to $\mathbb{F}_2[Y]$ and is related with $g(x)$ with regard to 2 when we use the notations as in the previous definition. We also observe that $e = 2$, $n = 4$, and $t = 2$.

In a broader context, we will use the following less robust formulations of the two theorems demonstrated by Ore. Their arguments are not included.

5. Proof

You may remember that

$$D_{K/\mathbb{Q}}(1, \theta, \dots, \theta^{n-1}) = (\text{Ind } \theta)^2 d_K = (-1)^{\binom{n}{2}} N_{K/\mathbb{Q}}(n\theta^{n-1}) = (-1)^{\frac{(n-1)(n-2)}{2}} n^n a^{n-1}.$$

Hence, a must be divided by any prime that divides $\text{Ind}(\theta)$.

$$\text{Ind}(\theta) = \prod_{i=1}^k p_i^{u_i} \prod_{j=1}^l q_j^{\frac{1}{2}[(n-1)(t_j-1)+m_j-1]},$$

depending on whether r_i is greater than zero or not, where u_i is equal to $n_i \sum_{j=1}^{\min\{r_i, s_i\}} p^{s-j}$ or 0. The formula for d_K can be quickly obtained by substituting the values of $\prod_{i=1}^k p_i^{s_i}$, $|a| = \prod_{j=1}^l q_j^{t_j}$ in equation (10), with $\text{Ind}(\theta)$ being the previously mentioned variable. Validation of the 1.3 Corollary. Premise first that $A_K = \mathbb{Z}[\mathbb{Q}]$. So, let's pretend that a isn't a squarefree integer, for example, $a = bc^2$ where c is greater than or equal to 2. An algebraic integer $\frac{\theta^{n-1}}{c} = \frac{bc}{\theta}$ can be deduced from the equality $\theta^n = a = bc^2$ which indicates that θ divides $(bc)^n$ in A_K . Consequently, θ divides bc . The result is that c will split the index $[A_K : \mathbb{Z}[\theta]]$. This inconsistency establishes that a is free of squares. The discriminant d_K of K is given by when a is squarefree, when we write the prime factorization of n as $\prod_{i=1}^k p_i^{s_i}$ and take r_i and v_i as in Theorem 1, and by

$$d_K = (-1)^{\frac{(n-1)(n-2)}{2}} \text{sgn}(a^{n-1}) \left(\prod_{i=1}^k p_i^{v_i} \right) |a|^{n-1}.$$

As may be shown from equation,

$$D_{K/\mathbb{Q}}(1, \theta, \theta^2, \dots, \theta^{n-1}) = (\text{Ind}(\theta))^2 d_K = (-1)^{\frac{(n-1)(n-2)}{2}} n^n a^{n-1}.$$

Looking at the two equations above, the set $\{1, \theta, \theta^2, \dots, \theta^{n-1}\}$ is an integral basis of K if and only if $n^n = \prod_{i=1}^k p_i^{v_i}$, that is, if and only if $\prod_{i=1}^k p_i^{n s_i} = \prod_{i=1}^k p_i^{v_i}$. Considered in light of v_i definition, the final equivalence is valid only in cases when r_i is less than or equal to zero, that is, when $p_i^2 \nmid (a^{p_i-1} - 1)$. We have now shown the corollary.

6. Conclusion

Finally, the connection between the arithmetic structure of number fields and the basic ideas of Algebraic Number Theory is strongly demonstrated by the analysis of discriminants and integral bases in pure number fields. Pure number fields, which are fundamental for the computational exploration of abstract theoretical concepts, are defined by simple equations of the type $K = \mathbb{Q}(na)$. In spite of their apparently elementary structure, these fields display extensive mathematical behavior, as shown in this paper, in particular with respect to the computation of discriminants and the generation of integral bases.

An important result of this study is the creation of a computationally tractable and explicit formula for the field K 's discriminant. The method improves accessibility and removes superfluous algebraic complexity by stating the discriminant in terms of prime power factorizations of the defining integer a and the degree n . Not only does this approach simplify computations, but it also clarifies the relationship between prime numbers and the field extension. Specifically, it elucidates the ramification phenomena by showing how specific primes affect the field's arithmetic structure by dividing the discriminant.

In addition, studying integral bases in this context further emphasizes their significance as a basic tool for OK-ring integer description. The findings demonstrate that the discriminant's behavior is directly related to the determination of an integral basis in pure number fields. The explicit discriminant formula is a powerful tool for checking if a set of elements forms an integral basis. Both theoretical knowledge and computing efficiency are greatly improved by this relationship, which is relevant to these areas of study.

The capacity to connect theoretical notions with practical computation is another significant contribution of this study. We lay a solid groundwork for future research in pure and applied mathematics by simplifying discriminant computations and better characterizing integral bases. These results are of special interest in computational number theory and related fields that use algebraic structures to solve problems, since effective algorithms in these areas rely on exact knowledge of field invariants.

Pure number fields, albeit structurally basic, provide deep insights into the nature of algebraic extensions, as this study broader points shown. Our knowledge of discriminants and integral bases is enhanced and new avenues for research are created by the explicit methodology and results presented here. Possible tasks include improving computing methods, investigating potential uses in domains like algebraic geometry and encryption, or expanding the approaches to cover broader classes of number fields.

In sum, the paper makes a substantial and thorough addition to the literature on pure number fields. It strengthens the theoretical depth and practical value of Algebraic Number Theory by providing a more computable and organized presentation of the integral bases and discriminants, which clarifies their purpose and lays a solid foundation for future research in the field.

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