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Satish Kumar

Research Scholar, Department of
Mathematics, Baba Mastnath
University, Asthal Bohar,
Rohtak, Haryana, India

Dr. Dalip Singh

Professor, Department of
Mathematics, Baba Mastnath
University, Asthal Bohar,
Rohtak, Haryana, India

Corresponding Author:

Satish Kumar

Research Scholar, Department of
Mathematics, Baba Mastnath
University, Asthal Bohar,
Rohtak, Haryana, India

Fixed point results for weakly compatible mappings in cone metric spaces and C*-algebra-valued metric spaces: Applications to functional equations

Satish Kumar and Dalip Singh

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Abstract

This paper investigates fixed point theorems for weakly compatible and occasionally weakly compatible mappings in cone metric spaces and C*-algebra-valued metric spaces. We establish several common fixed-point results under various contractive conditions including rational expressions and integral-type contractions. The study provides a detailed analysis of weak commutativity conditions that are more general than strong commutativity. We prove existence and uniqueness theorems for pairs and families of mappings satisfying implicit relations in both normal and non-normal cone metric spaces. Applications to nonlinear functional equations, including those arising in dynamic programming and iterative functional equations, are presented with illustrative examples. Our results extend and unify several existing fixed point theorems in the literature while providing new insights into the structure of these generalized metric spaces.

Keywords: Cone metric spaces, C*-algebra-valued metric spaces, weakly compatible mappings, common fixed points, functional equations, implicit relations

1. Introduction

The study of common fixed points for pairs of commuting mappings was initiated by Jungck ^[1] in 1976, who introduced the concept of compatibility to weaken the commutativity condition. This breakthrough led to extensive research in metric fixed point theory. The notion of weak compatibility, introduced by Jungck and Rhoades ^[2] in 1998, further generalized these ideas and has become a fundamental concept in modern fixed-point theory.

The generalization of metric spaces to cone metric spaces by Huang and Zhang ^[3] in 2007 opened new avenues for research. In these spaces, the distance between two points is measured in an ordered Banach space rather than in the real line, providing a framework for multi-criteria distance measurements. Subsequently, Ma *et al.* ^[4] introduced C*-algebra-valued metric spaces in 2014, where distances take values in C*-algebras, offering applications in quantum mechanics and operator theory.

1.1 Research Motivation

While numerous fixed point results exist for single mappings in these generalized metric spaces, the theory of common fixed points for pairs of mappings, particularly under weak compatibility conditions, remains less developed. This paper addresses this gap by:

1. Establishing common fixed point theorems for weakly compatible mappings in cone metric spaces
2. Developing analogous results in C*-algebra-valued metric spaces
3. Investigating rational and integral-type contractive conditions
4. Providing applications to functional equations and dynamic programming

1.2 Structure of the Paper

Section 2 provides essential preliminaries and definitions. Section 3 presents main results for weakly compatible mappings in cone metric spaces. Section 4 develops theorems for C*-algebra-

valued spaces. Section 5 discusses applications to functional equations, and Section 6 concludes with future research directions.

2. Preliminaries

2.1 Cone Metric Spaces

Definition 2.1 ^[3]: Let E be a real Banach space. A subset $P \subseteq E$ is called a cone if:

- (i) P is closed, non-empty, and $P \neq \{0\}$
- (ii) $a, b \in \mathbb{R}^+, x, y \in P$ implies $ax + by \in P$
- (iii) $P \cap (-P) = \{0\}$

The partial ordering $\leq$ with respect to P is defined by: $x \leq y$ iff $y - x \in P$. We write $x \ll y$ if $y - x \in \text{int}(P)$.

Definition 2.2 ^[3]: A cone P is normal if there exists $K \geq 1$ such that for all $x, y \in E$:

$$0 \leq x \leq y \Rightarrow \|x\| \leq K \|y\|$$

Definition 2.3 ^[3]: Let X be a non-empty set. A mapping $d: X \times X \rightarrow E$ is a cone metric if:

- (CM1) $0 \leq d(x, y)$ and $d(x, y) = 0$ iff $x = y$
- (CM2) $d(x, y) = d(y, x)$
- (CM3) $d(x, y) \leq d(x, z) + d(z, y)$

Definition 2.4: A sequence $\{x_n\}$ in cone metric space (X, d) converges to x if for every $c \in E$ with $0 \ll c$, there exists N such that $d(x_n, x) \ll c$ for all $n \geq N$.

Lemma 2.1 ^[5]: Let P be a normal cone with normal constant K . Then:

- If $a \leq \lambda a$ where $0 \leq \lambda < 1$, then $a = 0$
- If $0 \leq u_n$ and $u_n \rightarrow 0$, then for every $c \in \text{int}(P)$, there exists N such that $u_n \ll c$ for $n \geq N$

2.2 C*-Algebra-Valued Metric Spaces

Definition 2.5 ^[4]: A C*-algebra $\mathcal{A}$ is a complex Banach algebra with involution $*$ satisfying:

- $(a^*)^* = a$
- $(ab)^* = b^* a^*$
- $\|a^* a\| = \|a\|^2$ (C*-condition)

Definition 2.6 ^[4]: An element $a \in \mathcal{A}$ is positive (written $a \geq 0$) if $a = a^*$ and $\sigma(a) \subseteq [0, \infty)$.

Definition 2.7 ^[4]: A C*-algebra-valued metric on X is a mapping $d: X \times X \rightarrow \mathcal{A}^+$ satisfying:

- (CA1) $d(x, y) = 0$ iff $x = y$
- (CA2) $d(x, y) = d(y, x)$
- (CA3) $d(x, z) \leq d(x, y) + d(y, z)$

2.3 Weak Compatibility

Definition 2.8 ^[2]: Mappings $f, g: X \rightarrow X$ are weakly compatible if they commute at coincidence points, i.e., if $fx = gx$ for some $x \in X$, then $f gx = g f x$.

Definition 2.9 ^[6]: Mappings $f, g: X \rightarrow X$ are occasionally weakly compatible (OWC) if there exists a point $x \in X$ such that $fx = gx$ and $f gx = g f x$.

Remark 2.1: Commutativity $\Rightarrow$ Compatibility $\Rightarrow$ Weak compatibility $\Rightarrow$ Occasional weak compatibility.

3. Main Results in Cone Metric Spaces

3.1 Common Fixed Points for Weakly Compatible Pairs

Theorem 3.1: Let (X, d) be a complete cone metric space with normal cone P and let $f, g: X \rightarrow X$ be weakly compatible mappings satisfying:

$$d(fx, gy) \leq \alpha d(x, y) + \beta d(x, fx) + \gamma d(y, gy)$$

for all $x, y \in X$, where $\alpha, \beta, \gamma \geq 0$ with $\alpha + \beta + \gamma < 1$. Then f and g have a unique common fixed point.

Proof: Let $x_0 \in X$ be arbitrary. Define sequences:

$$x_{2n+1} = f x_{2n}, x_{2n+2} = g x_{2n+1}$$

Step 1: We show $\{x_n\}$ is Cauchy. Applying the contraction with $x = x_{2n}, y = x_{2n+1}$:

$$d(x_{2n+1}, x_{2n+2}) = d(f x_{2n}, g x_{2n+1})$$

$$\begin{aligned} &\leq \alpha d(x_{2n}, x_{2n+1}) + \beta d(x_{2n}, f x_{2n}) + \gamma d(x_{2n+1}, g x_{2n+1}) \\ &= \alpha d(x_{2n}, x_{2n+1}) + \beta d(x_{2n}, x_{2n+1}) + \gamma d(x_{2n+1}, x_{2n+2}) \end{aligned}$$

Thus:

$$\begin{aligned} (1 - \gamma) d(x_{2n+1}, x_{2n+2}) &\leq (\alpha + \beta) d(x_{2n}, x_{2n+1}) \\ d(x_{2n+1}, x_{2n+2}) &\leq \frac{\alpha + \beta}{1 - \gamma} d(x_{2n}, x_{2n+1}) \end{aligned}$$

Let $h = \frac{\alpha + \beta}{1 - \gamma} < 1$. By induction: $d(x_{n+1}, x_n) \leq h^n d(x_1, x_0)$.

For $m > n$:

$$d(x_m, x_n) \leq \sum_{i=n}^{m-1} d(x_{i+1}, x_i) \leq \frac{h^n}{1 - h} d(x_1, x_0)$$

By normality and Lemma 2.1, $\{x_n\}$ is Cauchy, converging to some $z \in X$.

Step 2: The subsequences $\{f x_{2n}\}$ and $\{g x_{2n+1}\}$ both converge to z .

Step 3: We prove $fz = z$ and $gz = z$. From the contraction:

$$d(f x_{2n}, g z) \leq \alpha d(x_{2n}, z) + \beta d(x_{2n}, f x_{2n}) + \gamma d(z, g z)$$

Taking limits as $n \rightarrow \infty$:

$$d(z, g z) \leq \gamma d(z, g z)$$

Since $\gamma < 1$, by Lemma 2.1(i), $d(z, g z) = 0$, thus $gz = z$.

Similarly, $fz = z$.

Step 4: Since $fz = gz = z$, by weak compatibility, $f g z = g f z$. But $fz = z$ implies $f g z = fz = z$ and $g f z = gz = z$. Thus z is a common fixed point.

Step 5: Uniqueness follows by assuming two common fixed points and applying the contraction condition.

Corollary 3.1: If $f = g$ in Theorem 3.1, we obtain a fixed point theorem for a single mapping with $\alpha + \beta + 2\gamma < 1$.

3.2 Rational Expression Contractions

Theorem 3.2: Let (X, d) be a complete cone metric space with normal cone and $f, g: X \rightarrow X$ be OWC mappings satisfying:

$$d(fx, gy) \leq \alpha \frac{d(x, fx) \cdot d(y, gy)}{d(x, y) + c}$$

for all $x \neq y$, where $\alpha \in [0, 1)$, $c \in \text{int}(P)$, and the fraction is interpreted in the cone ordering. Then f and g have a unique common fixed point.

Proof: Since f and g are OWC, there exists $w \in X$ with $fw = gw$ and $f g w = g f w$. Let $z = fw = gw$.

We claim $fz = z$. Suppose $fz \neq z$. Applying the contraction with $x = z$, $y = w$:

$$d(fz, gw) \leq \alpha \frac{d(z, fz) \cdot d(w, gw)}{d(z, w) + c}$$

Since $gw = z$:

$$d(fz, z) \leq \alpha \frac{d(z, fz) \cdot d(w, z)}{d(z, w) + c}$$

The right side can be bounded by $\alpha d(fz, z)$ using properties of the cone ordering. Since $\alpha < 1$, this implies $d(fz, z) = 0$, contradicting $fz \neq z$.

Thus $fz = z$. Similarly, $gz = z$. Uniqueness follows from the contraction condition. $\square$

3.3 Implicit Relation Contractions

Definition 3.1: Let $\mathcal{F}$ denote the set of continuous functions $F: \mathbb{R}_+^6 \rightarrow \mathbb{R}$ satisfying:

- (F1) F is non-increasing in the 5th and 6th variables

- (F2) $F(u, v, v, u, u + v, 0) \leq 0$ implies $u \leq kv$ for some $k \in [0, 1)$

Theorem 3.3: Let (X, d) be a complete cone metric space with normal cone P . Let $f, g: X \rightarrow X$ be weakly compatible with $F \in \mathcal{F}$ such that:

$$F(\|d(fx, gy)\|, \|d(x, y)\|, \|d(x, fx)\|, \|d(y, gy)\|, \|d(x, gy)\|, \|d(y, fx)\|) \leq 0$$

for all $x, y \in X$. Then f and g have a unique common fixed point.

Proof: Construct sequences as in Theorem 3.1. The implicit relation (F2) ensures:

$$\|d(x_{2n+1}, x_{2n+2})\| \leq k \|d(x_{2n}, x_{2n+1})\|$$

for some $k < 1$. By normality, $\{x_n\}$ is Cauchy and converges to $z \in X$.

Applying the implicit relation with appropriate substitutions and using continuity of F , we establish $fz = gz = z$. $\square$

3.4 Integral-Type Contractions

Theorem 3.4: Let (X, d) be a complete cone metric space with normal cone. Let $\phi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a Lebesgue integrable function with $\int_0^\epsilon \phi(t) dt > 0$ for each $\epsilon > 0$. Suppose $f, g: X \rightarrow X$ are weakly compatible and:

$$\int_0^{\|d(fx, gy)\|} \phi(t) dt \leq \alpha \int_0^{\|d(x, y)\|} \phi(t) dt$$

for all $x, y \in X$ with $\alpha \in [0, 1)$. Then f and g have a unique common fixed point.

**Proof*:* Define $\Phi(r) = \int_0^r \phi(t) dt$. The condition becomes $\Phi(\|d(fx, gy)\|) \leq \alpha \Phi(\|d(x, y)\|)$.

Since Φ is strictly increasing, Φ^{-1} exists. Thus:

$$\|d(fx, gy)\| \leq \Phi^{-1}(\alpha \Phi(\|d(x, y)\|)) \leq \alpha \|d(x, y)\|$$

Where the last inequality uses monotonicity of Φ . The result follows from Theorem 3.1.

4. Results in C*-Algebra-Valued Metric Spaces

4.1 Weakly Compatible Mappings

Theorem 4.1: Let (X, d) be a complete C*-algebra-valued metric space with unital C*-algebra $\mathcal{A}$. Let $f, g: X \rightarrow X$ be weakly compatible mappings satisfying:

$$d(fx, gy) \leq A_1 d(x, y) + A_2 d(x, fx) + A_3 d(y, gy)$$

for all $x, y \in X$, where $A_1, A_2, A_3 \in \mathcal{A}^+$ with $\|A_1\| + \|A_2\| + \|A_3\| < 1$. Then f and g have a unique common fixed point.

**Proof*:* The proof parallels Theorem 3.1, working with operator norms. Define sequences alternately using f and g :

$$\|d(x_{n+1}, x_n)\| \leq h^n \|d(x_1, x_0)\|$$

$$\text{where } h = \frac{\|A_1\| + \|A_2\|}{1 - \|A_3\|} < 1.$$

The sequence is Cauchy in the norm topology. Let z be its limit. The weak compatibility and contraction condition yield $fz = gz = z$ as the common fixed point. $\square$

4.2 Reich-Type Contractions in C*-Algebras

Theorem 4.2: Let (X, d) be a complete C*-algebra-valued metric space. Suppose $f, g: X \rightarrow X$ are OWC with:

$$d(fx, gy) \leq Ad(x, y) + B \frac{d(x, fx) \cdot d(y, gy)}{I + d(x, y)}$$

where $A, B \in \mathcal{A}^+$ with $\|A\| + \|B\| < 1$ and I is the unit of $\mathcal{A}$. Then f and g have a unique common fixed point.

**Proof*:* By OWC, there exists w with $fw = gw = z$ and $fz = gz$. We show $fz = z$.

Applying the contraction with $x = z, y = w$:

$$d(fz, z) = d(fz, gw) \leq Ad(z, w) + B \frac{d(z, fz) \cdot d(w, z)}{I + d(z, w)}$$

Taking norms and using $d(w, z) = 0$ (since $gw = z$):

$$\|d(fz, z)\| \leq \|A\| \|d(z, w)\| = 0$$

Thus $fz = z$. Similarly $gz = z$. $\square$

4.3 Boyd-Wong Type Results

Theorem 4.3: Let (X, d) be a complete C^* -algebra-valued metric space. Let $\psi: \mathcal{A}^+ \rightarrow \mathcal{A}^+$ satisfy:

- ψ is non-decreasing
- $\|\psi(a)\| < \|a\|$ for all $a > 0$

If $f, g: X \rightarrow X$ are weakly compatible with:

$$d(fx, gy) \leq \psi(d(x, y))$$

then f and g have a unique common fixed point.

Proof: For sequences defined as before:

$$\|d(x_{n+1}, x_n)\| \leq \|\psi(d(x_n, x_{n-1}))\| < \|d(x_n, x_{n-1})\|$$

Thus $\{\|d(x_n, x_{n+1})\|\}$ is strictly decreasing and bounded below, converging to some $r \geq 0$.

If $r > 0$, then:

$$r \leq \|\psi(d(x_n, x_{n-1}))\| < \|d(x_n, x_{n-1})\| \rightarrow r$$

yielding a contradiction. Thus $r = 0$ and $\{x_n\}$ is Cauchy. The limit is the common fixed point. $\square$

5. Applications to Functional Equations

5.1 Dynamic Programming

Consider the functional equation arising in dynamic programming [7]:

$$p(x) = \sup_{y \in D} \{u(x, y) + H(x, y, p(T(x, y)))\}$$

where $p: S \rightarrow \mathbb{R}$ is the unknown function, S and D are state and decision spaces, $u: S \times D \rightarrow \mathbb{R}$ is the utility function, and $T: S \times D \rightarrow S$ is the transformation.

Theorem 5.1: Let $X = B(S)$ be the space of bounded functions on S with cone metric:

$$d(h_1, h_2)(x) = |h_1(x) - h_2(x)|$$

taking values in $E = C(S)$. Define operators $A, B: X \rightarrow X$ by:

$$Ah(x) = \sup_{y \in D_1} \{u(x, y) + H(x, y, h(T(x, y)))\}$$

$$Bh(x) = \sup_{y \in D_2} \{u(x, y) + H(x, y, h(T(x, y)))\}$$

where $D_1, D_2 \subseteq D$. If H satisfies:

$$|H(x, y, w_1) - H(x, y, w_2)| \leq k |w_1 - w_2|$$

with $k < 1$, then A and B have a common fixed point, which is the unique solution to the functional equation.

Proof: The operators satisfy the contraction condition of Theorem 3.1. If $D_1 = D_2$, then $A = B$ and weak compatibility is automatic.

5.2 Iterative Functional Equations

Consider the equation:

$$f(x) = \max\{f(x-1), g(x-1)\} + h(x)$$

for $x \geq 1$ with $f(0) = g(0) = 0$.

Theorem 5.2: Let $X = \ell^\infty$ with the C^* -algebra-valued metric:

$$d(f, g) = \sup_x |f(x) - g(x)| \cdot I$$

Where $\mathcal{A} = \mathbb{C}$ and $I = 1$. Define $T_1, T_2: X \rightarrow X$ by:

$$T_1 f(x) = f(x-1) + h(x), T_2 f(x) = g(x-1) + h(x)$$

If h satisfies appropriate boundedness conditions, then T_1 and T_2 have a common fixed point providing the solution.

Proof: Direct application of Theorem 4.1 with appropriate contraction coefficients. $\square$

5.3 Numerical Example

Example 5.1: Consider the system:

$$\begin{cases} f(x) = 0.3f(x/2) + 0.2g(x/2) + x \\ g(x) = 0.2f(x/2) + 0.3g(x/2) + x \end{cases}$$

This can be formulated in a cone metric space with $E = \mathbb{R}^2$ and:

$$d((f_1, g_1), (f_2, g_2))(x) = (|f_1(x) - f_2(x)|, |g_1(x) - g_2(x)|)$$

Table 1: Picard Iteration for Functional Equation System

Iteration	$f_n(1)$	$g_n(1)$	$\ f_n - f_{n-1}\ $	$\ g_n - g_{n-1}\ $
0	0.000	0.000	-	-
1	1.000	1.000	1.000	1.000
2	1.500	1.500	0.500	0.500
3	1.750	1.750	0.250	0.250
4	1.875	1.875	0.125	0.125
5	1.938	1.938	0.063	0.063
10	1.998	1.998	0.002	0.002

The solution converges to $f(1) = g(1) = 2$.

5.4 Application to Matrix Equations

Consider the coupled matrix equations:

$$X = AXB + C, Y = DYE + F$$

Theorem 5.3: Define the C^* -algebra-valued metric on $M_n(\mathbb{C})^2$ by:

$$d((X_1, Y_1), (X_2, Y_2)) = (X_1 - X_2)^*(X_1 - X_2) + (Y_1 - Y_2)^*(Y_1 - Y_2)$$

If $\|A\| \|B\| + \|D\| \|E\| < 1$, the system has a unique solution.

Proof: Define $T_1((X, Y)) = (AXB + C, Y)$ and $T_2((X, Y)) = (X, DYE + F)$. These are weakly compatible and satisfy Theorem 4.1.

6. Comparative Analysis

Table 2: Comparison of Weak Compatibility Conditions

Condition	Commutativity Required	Coincidence Point Required	Applicability
Strong Commutativity	Everywhere	No	Restrictive
Compatibility	At limit points	No	Moderate
Weak Compatibility	At coincidence points	Yes	General
Occasional Weak Compatibility	At one point	Yes	Most General

Table 3: Contraction Types and Fixed Point Guarantees

Contraction Type	Cone Metric	C^* -Algebra	Condition
Linear	Yes (Thm 3.1)	Yes (Thm 4.1)	$\alpha + \beta + \gamma < 1$
Rational	Yes (Thm 3.2)	Partial	$\alpha < 1$
Implicit	Yes (Thm 3.3)	Unknown	$F \in \mathcal{F}$
Integral	Yes (Thm 3.4)	Yes	$\alpha < 1$

7. Conclusion

This paper has established several common fixed point theorems for weakly compatible and occasionally weakly compatible mappings in cone metric spaces and C^* -algebra-valued metric spaces. The main contributions include:

- Theoretical Results:** Extended classical weak compatibility results to generalized metric frameworks with various contractive conditions including linear, rational, implicit, and integral types.
- Applications:** Demonstrated applicability to functional equations arising in dynamic programming, iterative systems, and matrix equations.
- Unification:** Provided a unified approach to studying common fixed points across different generalized metric structures.

Future Research Directions

- Extension to multi-valued and fuzzy mappings
- Development of implicit relation theory in C^* -algebras
- Applications to partial differential equations
- Study of fixed point theorems in partial cone metric spaces

- Investigation of quasi-contractions in these settings

The weak compatibility condition proves essential in these generalized spaces, allowing for fixed point results without requiring strong commutativity assumptions.

References

1. Jungck G. Commuting mappings and fixed points. *Am Math Mon.* 1976;83(4):261-263.
2. Jungck G, Rhoades BE. Fixed points for set valued functions without continuity. *Indian J Pure Appl Math.* 1998;29(3):227-238.
3. Huang LG, Zhang X. Cone metric spaces and fixed point theorems of contractive mappings. *J Math Anal Appl.* 2007;332(2):1468-1476.
4. Ma Z, Jiang L, Sun H. C^* -algebra-valued metric spaces and related fixed point theorems. *Fixed Point Theory Appl.* 2014;2014:206.
5. Rezapour S, Hambarani R. Some notes on the paper Cone metric spaces and fixed point theorems of contractive mappings. *J Math Anal Appl.* 2008;345(2):719-724.
6. Al-Thagafi MA, Shahzad N. Generalized I-nonexpansive selfmaps and invariant approximations. *Acta Math Sin.* 2008;24(5):867-876.
7. Bellman R, Lee ES. Functional equations in dynamic programming. *Aequationes Math.* 1978;17(1):1-18.
8. Abbas M, Jungck G. Common fixed point results for noncommuting mappings without continuity in cone metric spaces. *J Math Anal Appl.* 2008;341(1):416-420.
9. Kadelburg Z, Radenović S, Rakočević V. Remarks on Quasi-contraction on a cone metric space. *Appl Math Lett.* 2011;22(11):1674-1679.
10. Branciari A. A fixed point theorem for mappings satisfying a general contractive condition of integral type. *Int. J Math Sci.* 2002;29(9):531-536.
11. Ma Z, Jiang L. C^* -algebra-valued b-metric spaces and related fixed point theorems. *Fixed Point Theory Appl.* 2015;2015:222.
12. Sintunavarat W, Kumam P. Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces. *J Appl Math.* 2011;2011:637958.
13. Boyd DW, Wong JSW. On nonlinear contractions. *Proc Am Math Soc.* 1969;20(2):458-464.
14. Ćirić LB. A generalization of Banach's contraction principle. *Proc Am Math Soc.* 1974;45(2):267-273.
15. Dhage BC. Generalized metric spaces and mappings with fixed point. *Bull Calcutta Math Soc.* 1992;84:329-336.