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Fixed point theorems for multivalued mappings in G- and G-cone metric spaces using G-Hausdorff contractions

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Abstract

This paper establishes two general fixed-point theorems for multivalued contractive-type maps in G-metric and G-cone metric spaces. A G-Hausdorff contraction condition is generalized to extend the classical results. Nadler's theorem in the setting of an advanced metric space endowed with higher-order and cone-valued metrics. The framework merges previous findings from the more general extended b-metric and G-metric space theory by considering explicit cone-based inequalities combined with rigorous data-based convergence. From this study, one can see that the fixed-point theorem in the G-cone metric context generalizes the corresponding one in the G-metric space, which shows the more powerful ability of cone-valued analysis. Furthermore, the compactness and upper semi-continuity conditions for multivalued mappings are characterized in this general setting, which allows the results to be applied to more general nonlinear systems. In general, the results provide a more general, analytically solid platform for the fixed point theory in abstract metric-like setting and they promote the development of functional and nonlinear analysis.

Keywords: Multivalued mappings, fixed point theorems, G-metric spaces, G-cone metric spaces, generalized Hausdorff contraction, cone metric space, contractive mappings

1. Introduction

Fixed-point theory is a core component of nonlinear analysis, which has been found to be a useful tool in various applications, including differential equations and integral equations, control systems, economics, and computer science. The classical Banach contraction principle is an important finding in this field that ensures that the unique fixed point of self-mappings of complete metric spaces is a contraction. Since its formulation, this principle has spawned an enormous literature to generalize fixed-point results in terms of weakening the assumption made on either the metric structure or the contractive condition or the character of the mappings one makes. One significant advance in this direction was achieved by Nadler, who generalized the contraction principle of Banach to multivalued mappings using the Hausdorff metric. This extension was feasible for studying the fixed points of set-valued operators, which naturally occur in optimization problems, differential inclusions, and equilibrium models. After this, it has been noted that much interest has been given to fixed point theory in generalized metric spaces, in which the classical metric axioms are relaxed to facilitate wider classes of nonlinear problems.

Among these generalizations is the definition of G-metric spaces, which was proposed by Mustafa and Sims, in which the distance function takes three variables rather than two. It is more flexible and may be interpreted geometrically richer than standard metric spaces. Many fixed-point findings, such as Banach-type and cyclic contraction theorems, have been successfully extended to G-metric spaces and their variants [7, 17, 21]. Huang and Zhang independently proposed cone metric spaces, where the distance is a value of an ordered Banach space, with a cone. Cone metric spaces have also been applied to solve problems and systems in the form of vectors controlled by partial orderings [5, 18, 22]. The intersection of the two structures results in the presentation of G-cone metric spaces, which brings together the

ideas of G-metric spaces and cone metric spaces on one platform. The distance of a G-cone metric space is a function of three variables and is valued in a cone, which means that both scalar-valued and concurrently vector-valued fixed-point problems can be treated. This single structure has become the focus of increasing attention because of its capacity to make generalizations about a variety of known results and offers a natural framework for partially ordered nonlinear systems ^[1, 3, 23].

The fixed-point theorems of multivalued mappings have been widely studied in recent years in many generalized metric-type spaces, such as b-metric spaces, partial metric spaces, and cone metric spaces. Other authors have generalized Nadler-type contraction results using generalized Hausdorff and graph-based methods ^[6, 8, 15, 24]. However, many of these studies are confined only to scalar-valued spaces or are heavily based on abstract set-theoretic arguments, which tend to mask the convergence mechanism at work. Furthermore, only a small number of studies have discussed multivalued contractions in the common context of G-metric and G-cone metric spaces in terms of constructive methods designed through overt recursive inequalities. Based on these findings, this study attempts to develop new fixed point theorems for multivalued contractive-type mappings in both G-metric and G-cone metric spaces through a generalized G-Hausdorff contraction condition. We generalize some classical findings of Nadler, Mustafa-Sims, and Huang-Zhang ^[1, 5, 7, 25] to compact-valued multivalued operators and cone induced partial orderings. In contrast to the use of purely abstract methods, the suggested algorithm builds sequences as a result of multivalued mapping and examines the convergence of the latter using explicit contractive estimates, producing transparent and verifiable proofs.

2. Preliminaries

Definition 2.1 (G-Metric Space; Reformulated from Mustafa and Sims, 2006)

Let M be a nonempty set and the function $d: M \times M \times M \rightarrow [0, \infty)$ be defined such that for all $p, q, r, s \in M$, The following conditions are fulfilled:

- $d(p, p, q) = 0$ if and only if $p = q$.
- $d(p, p, q) > 0$ whenever $p \neq q$,
- $d(p, p, q) \leq d(p, q, r)$,
- $d(p, q, r)$ remains unchanged under any permutation of p, q, r ,
- $d(p, q, r) \leq d(p, s, s) + d(s, q, r)$.

A structure (M, d) satisfying the above is termed a G-metric space.

Definition 2.2 (Convergence and Completeness in G-Metric Spaces)

A sequence $\{p_n\} \subset M$ is:

- G-Cauchy if $\lim_{m,n \rightarrow \infty} d(p_m, p_n, p_k) = 0$ for any k ,
- G-convergent to $p \in M$ if $\lim_{n \rightarrow \infty} d(p_n, p_n, p) = 0$.

The space (M, d) is said to be complete if every G-Cauchy sequence converges to a point in M .

Definition 2.3 (Cone in a Normed Vector Space)

Let V be a real Banach space and $C \subset V$ be a nontrivial closed subset. Then C is a cone if:

- C is closed, nonempty, and $C \neq \{0\}$,
- $\lambda x + \mu y \in C$ for all $x, y \in C$ and $\lambda, \mu \geq 0$,
- If $x \in C$ and $-x \in C$, then $x = 0$.

A partial ordering on V is then described by: $x \leq y \Leftrightarrow y - x \in C$.

Definition 2.4 (Multivalued Mappings)

A mapping $F: M \rightarrow 2^M \setminus \{\emptyset\}$ is termed multivalued if for each $p \in M$, $F(p)$ is a nonempty subset of M . With reference to fixed-point theory using the G-Hausdorff distance, values of F are usually assumed to be compact.

Definition 2.5 (Generalized G-Hausdorff Distance)

Let $A, B \subset M$ be compact sets that are nonempty in a G-metric (or G-cone metric) space. Then the G-Hausdorff distance $H_d(A, B)$ is defined as:

$$H_d(A, B) = \sup_{a \in A} \inf_{b \in B} d(a, b, b)$$

In cone-based structures, the distance d is evaluated with respect to the cone-induced partial order in V .

Definition 2.6 (G-Cone Metric Space)

Suppose that V is a Banach space and that the $C \subset V$ satisfying:

- $C + C \subset C$, and $\alpha C \subset C$ for all $\alpha \geq 0$,
- $C \cap (-C) = \{0\}$, where 0 is the zero element in V .

Define the order $u \leq v \Leftrightarrow v - u \in C$. A function $d: M \times M \times M \rightarrow V$ is a G-cone metric if:

- It satisfies all G-metric conditions (adapted with respect to the cone order),
- The image of d lies in the cone C .

Completeness and convergence are defined analogously, using the norm and partial ordering from V .

Definition 2.7 (Modified G-Hausdorff Metric for Set-Valued Maps)

Let $F: Y \rightarrow \mathcal{K}(Y)$ be a multivalued operator, where $\mathcal{K}(Y)$ denotes the set of all nonempty compact subsets of Y . The generalized G-Hausdorff distance $H_G(A, B, C)$ for three such sets $A, B, C \subset Y$ is given by:

$$H_G(A, B, C) = \max \left\{ \sup_{a \in A} \inf_{b \in B, c \in C} G(a, b, c), \sup_{b \in B} \inf_{a \in A, c \in C} G(b, a, c), \sup_{c \in C} \inf_{a \in A, b \in B} G(c, a, b) \right\}$$

In the context of cone metric structures, this expression takes values in a cone $Q \subset E$, and comparisons use the induced partial order $\leq$.

Lemma 2.8 (Convergence of Geometrically Contractive Sequences)

Let $\{y_n\} \subset Y$ be a sequence satisfying the inequality:

$$G(y_n, y_{n+1}, y_{n+1}) \leq \lambda^n G(y_0, y_1, y_1) \text{ for some } 0 < \lambda < 1$$

Then, the sequence $\{y_n\}$ is G-Cauchy.

Lemma 2.9 (Uniqueness of G-Limit in Cauchy Sequences)

Suppose $\{y_n\}$ is a G-Cauchy sequence that converges to a point $y^* \in Y$. Then, any convergent subsequence of $\{y_n\}$ also converges to y^* , and

$$\lim_{n \rightarrow \infty} G(y_n, y^*, y^*) = 0$$

Theorem 2.10 (Mustafa-Sims Type Fixed Point Result in G-Metric Spaces)

Let $S: Y \rightarrow Y$ be a single-valued mapping such that, given (Y, G) as a full G-metric space,

$$G(Su, Sv, Sv) \leq \beta \cdot G(u, v, v) \quad \forall u, v \in Y, 0 < \beta < 1$$

Then S possesses a unique fixed point in Y .

Theorem 2.11 (Huang-Zhang Fixed Point in Cone Metric Spaces)

Let $S: Y \rightarrow Y$ meet the following conditions, and let (Y, d) be a full cone metric space equipped with a normal cone $Q \subset E$.

$$d(Su, Sv) \leq \gamma \cdot d(u, v), \text{ for some } 0 < \gamma < 1$$

Then S admits a unique fixed point in Y .

Theorem 2.12 (The Fixed Point Theorem for Cone Metric Spaces with Multivalued Contractions)

Consider a complete cone metric space (Y, d) , and let $F: Y \rightarrow \mathcal{CB}(Y)$ be a multivalued operator such that:

$$H_d(Fu, Fv) \leq \theta \cdot d(u, v), \text{ for all } u, v \in Y \text{ and } 0 < \theta < 1$$

Then F has at least one fixed point, i.e., there exists $y \in Y$ with $y \in F(y)$.

(This result generalizes Nadler's theorem to cone metric spaces.)

Lemma 2.13 (Fixed Point Containment via Upper Semi-Continuity)

If a multivalued operator $F: Y \rightarrow \mathcal{CB}(Y)$ in a G-metric or G-cone metric space has compact values and is upper semi-continuous, and if a fixed point exists, then that point belongs to its image: $y^* \in F(y^*)$.

(This ensures the constructed sequence's limit is a valid fixed point.)

Lemma 2.14 (Compact Multivalued Mappings)

Let $F: Y \rightarrow \mathcal{CB}(Y)$ be a multivalued operator with compact values. If Y is complete under a G-metric (or G-cone metric), then the graph of F , defined as:

$$\text{Graph}(F) = \{(x, y) \in Y \times Y \mid y \in F(x)\}$$

is closed in the topology induced by the G-metric.

The first part of this paper prepares a solid background for the fixed point results by introducing and investigating fundamental concepts in G-metric and G-cone metric theories. Definition 2.1 The notion of a G-metric space introduced by Mustafa and Sims by a metric with three arguments. This is an extension of the classical metrics in which richer geometric interpretations are allowed, and is essential from the point of view of nonlinear and multivalued analysis. Definition 2.2 defines the concepts of a G-Cauchy sequence and G-convergence that generalize the traditional notions of convergence and completeness. These are crucial for the fixed point analysis where convergence of sequences arising from PMEP contractive mappings constitutes a major proof

technique. Summarizing this concept, Definition 2.3 introduces cones in Banach spaces which then induces a partial ordering, and this partial ordering will play a crucial role in the structure of G-cone metric spaces. Definition 2.7 then represents a synthesis of these ideas which defines G-cone metric spaces in terms of the metric being a G-metric with its image in a cone. This two-sided system is particularly handy to study vector-valued or partially ordered systems. Nevertheless, as we demonstrate in the following sections, the multivalued mappings and the G-Hausdorff distance provided in Definitions 2.5 and 2.6 serve to extend the fixed point results to set-valued functions. The convergence results are strengthened by Lemma 2.9, which states that geographically contractive sequences are G-Cauchy, and Lemma 2.10, which guarantees the uniqueness of limits in such sequences. These estimates guarantee stability in the convergence analysis. Theorems 2.11, 2.12 and 2.13 give essential fixed point theorems for single valued and multivalued mappings in G-metric and G-cone metric spaces. Theorem 2.13 is a generalization of Nadler's result on cone metric spaces. Lastly, we have the following Lemma 2.14, which enforces a fixed point to be in the same set, given by compactness of a graph in Lemma 2.15. Combined these classical facts provide a theoretical foundation for the advanced fixed point theorems presented later in the paper.

3. Main Results

Theorem 3.1: Let (Y, d_G) be a complete G-metric space where the G-metric is symmetric. Suppose $\mathcal{F}: Y \rightarrow \mathcal{K}(Y)$ is a multivalued function assigning to each point a nonempty compact subset of Y . Assume that for all $a, b, c \in Y$, the following contraction condition holds:

$$H_G(\mathcal{F}a, \mathcal{F}b, \mathcal{F}c) \leq \theta \cdot d_G(a, b, c)$$

where $\theta \in [0, 1)$ and H_G represents the G-Hausdorff metric. Then there exists a point $y^* \in Y$ such that $y^* \in \mathcal{F}(y^*)$; i.e., y^* is a fixed point of $\mathcal{F}$.

Proof

Choose an arbitrary $y_0 \in Y$. Since $\mathcal{F}(y_0)$ is nonempty and compact, select $y_1 \in \mathcal{F}(y_0)$.

Inductively define a sequence $\{y_n\} \subset Y$ such that $y_{n+1} \in \mathcal{F}(y_n)$ for each $n \in \mathbb{N}$. For a small tolerance $\delta_n := \frac{1}{n+1}$, the compactness of $\mathcal{F}(y_n)$ guarantees the existence of such y_{n+1} satisfying:

$$d_G(y_{n+1}, y_n, y_n) \leq H_G(\mathcal{F}y_n, \mathcal{F}y_{n-1}, \mathcal{F}y_{n-1}) + \delta_n$$

By the contraction assumption:

$$H_G(\mathcal{F}y_n, \mathcal{F}y_{n-1}, \mathcal{F}y_{n-1}) \leq \theta \cdot d_G(y_n, y_{n-1}, y_{n-1})$$

Combining gives

$$d_G(y_{n+1}, y_n, y_n) \leq \theta \cdot d_G(y_n, y_{n-1}, y_{n-1}) + \delta_n$$

Define $s_n := d_G(y_n, y_{n-1}, y_{n-1})$. Then:

$$s_{n+1} \leq \theta s_n + \delta_n$$

Applying recursive estimation

$$s_{n+1} \leq \theta^n s_1 + \sum_{j=1}^n \theta^{n-j} \delta_j$$

As $0 \leq \theta < 1$ and $\delta_j = \frac{1}{j+1} \rightarrow 0$, the sum converges. Therefore, $s_n \rightarrow 0$ as $n \rightarrow \infty$, implying:

$$d_G(y_n, y_{n-1}, y_{n-1}) \rightarrow 0$$

Using the triangle-type inequality for G-metric

$$d_G(y_n, y_m, y_m) \leq \sum_{k=m+1}^n d_G(y_k, y_{k-1}, y_{k-1}) = \sum_{k=m+1}^n s_k$$

Since $s_k \rightarrow 0$, for any $\varepsilon > 0$, there exists N such that for all $k \geq N$, $s_k < \frac{\varepsilon}{n-m+1}$. Thus:

$$d_G(y_n, y_m, y_m) < \varepsilon$$

So, $\{y_n\}$ is G-Cauchy. The completeness of (Y, d_G) ensures the existence of a limit point $y^* \in Y$ such that $y_n \rightarrow y^*$ in G-metric. Assume for contradiction that $y^* \notin \mathcal{F}(y^*)$. Since $y_{n+1} \in \mathcal{F}(y_n)$ and $y_n \rightarrow y^*$, we apply the contractive condition:

$$H_G(\mathcal{F}y_n, \mathcal{F}(y^*), \mathcal{F}(y^*)) \leq \theta \cdot d_G(y_n, y^*, y^*) \rightarrow 0$$

Then:

$$\lim_{n \rightarrow \infty} H_G(\mathcal{F}y_n, \mathcal{F}(y^*), \mathcal{F}(y^*)) = 0$$

So, for each $y_{n+1} \in \mathcal{F}(y_n)$, we have:

$$\inf_{z \in \mathcal{F}(y^*)} d_G(y_{n+1}, z, z) \rightarrow 0$$

This implies the existence of $z_n \in \mathcal{F}(y^*)$ such that:

$$d_G(y_{n+1}, z_n, z_n) \rightarrow 0$$

Since $y_{n+1} \rightarrow y^*$ and $z_n \rightarrow y^*$ by continuity of d_G , and $\mathcal{F}(y^*)$ is compact, it follows that $y^* \in \mathcal{F}(y^*)$, contradicting our assumption.

Thus, the contradiction implies:

$$y^* \in \mathcal{F}(y^*)$$

Hence, y^* is a fixed point of $\mathcal{F}$.

The existence of fixed points of multivalued contractive mappings in full symmetric G-metric spaces is demonstrated by Theorem 3.1. With the aid of a generalized G-Hausdorff contraction condition, the theorem establishes a sequence $\{y_n\}$ where each point is taken from the image of the next one under the multivalued map F . By comparing to a recursive inequality with an additive error term $\delta_n = 1/(n+1)$ it is proved that the sequence satisfies a contractive estimate and then converges to a limit point y^* . The completeness of the G-metric space and the compactness of the set-valued mapping serve as the foundation for the convergence proof. To confirm that y^* is, in fact, the fixed point, it is shown to belong to $F(y^*)$ by way of contradiction. Our result generalizes Nadler's fixed-point theorem to multivalued mappings in generalized metric spaces and gives an effective and meticulous method for studying multivalued mappings.

Corollary 3.1 (Existence of Fixed Point in Complete G-Metric Framework)

Statement: Let (M, d_G) be a complete G-metric space where the metric is symmetric. Consider a multivalued operator $\mathcal{F}: M \rightarrow \mathcal{K}(M)$, where $\mathcal{K}(M)$ represents the collection of all compact nonempty subsets of M . Suppose that $\mathcal{F}$ satisfies the generalized contraction criterion:

$$H_G(\mathcal{F}(u), \mathcal{F}(v), \mathcal{F}(w)) \leq \mu \cdot d_G(u, v, w), \forall u, v, w \in M$$

for some constant $\mu \in [0, 1)$. Then, there exists a point $u^* \in M$ such that $u^* \in \mathcal{F}(u^*)$, i.e., u^* is a fixed point of $\mathcal{F}$.

This result is a direct application of Theorem 3.1. Given the assumptions—namely, the completeness of the G-metric space, the compactness and non-emptiness of $\mathcal{F}(u)$ for all $u \in M$, and the Hausdorff-type contractive condition with coefficient $\mu \in [0, 1)$ —all prerequisites of the theorem are met. Thus, by the previously established convergence process, a fixed point u^* exists such that $u^* \in \mathcal{F}(u^*)$.

Corollary 3.2 (Convergence of Generated Sequence to a Fixed Point)

Let (X, G) be a complete symmetric G-metric space, and suppose $\mathcal{T}: X \rightarrow \mathcal{K}(X)$ is a multivalued mapping such that for all $x, y, z \in X$, the following contractive condition holds:

$$H_G(\mathcal{T}(x), \mathcal{T}(y), \mathcal{T}(z)) \leq \gamma \cdot G(x, y, z)$$

for some constant $\gamma \in [0, 1)$. Assume an iterative sequence $\{x_n\}$ is constructed beginning from an arbitrary point $x_0 \in X$, such that each $x_{n+1} \in \mathcal{T}(x_n)$ satisfies:

$$G(x_{n+1}, x_n, x_n) \leq \gamma \cdot G(x_n, x_{n-1}, x_{n-1}) + \frac{1}{n+1}$$

Then, the sequence $\{x_n\}$ converges to a limit point $x^* \in X$, and this point satisfies the fixed point condition:

$$x^* \in \mathcal{T}(x^*)$$

The recursive inequality establishes a decay pattern in the distances $G(x_{n+1}, x_n, x_n)$, showing that the sequence contracts over time. Because $\gamma < 1$ and the additive term $\frac{1}{n+1} \rightarrow 0$, the sequence is proven to be G-Cauchy. The completeness of (X, G) ensures

that the sequence converges to some limit point $x^* \in X$. Moreover, since the values of $\mathcal{T}$ are assumed to be compact, and the sequence $\{x_n\}$ is drawn from successive images of $\mathcal{T}$, the limit x^* must lie in the closure of $\mathcal{T}(x^*)$. Hence, $x^* \in \mathcal{T}(x^*)$, confirming it as a fixed point.

Example (Illustration of the Fixed Point Existence)

Let $M = [0,1]$ and define the G -metric d_G as:

$$d_G(a, b, c) = |a - b| + |b - c|, \forall a, b, c \in M$$

Define a multivalued operator $\mathcal{F}: M \rightarrow \mathcal{K}(M)$ as:

$$\mathcal{F}(x) = \left[\frac{x}{2}, 1 \right]$$

That is, for any $x \in [0,1]$, $\mathcal{F}(x)$ is the closed interval from $x/2$ to 1. We verify the assumptions:

- $\mathcal{F}(x)$ is nonempty and compact for every $x \in [0,1]$.
- For all $x, y, z \in [0,1]$, the G -Hausdorff distance between the sets $\mathcal{F}(x), \mathcal{F}(y), \mathcal{F}(z)$ satisfies:

$$H_G(\mathcal{F}(x), \mathcal{F}(y), \mathcal{F}(z)) \leq \frac{1}{2} \cdot d_G(x, y, z)$$

Construct a sequence $\{x_n\}$ with $x_0 = 1$:

- $x_1 \in [1/2, 1]$, choose $x_1 = 3/4$
- $x_2 \in [3/8, 1]$, choose $x_2 = 5/8$
- $x_3 \in [5/16, 1]$, choose $x_3 = 7/16$

This sequence monotonically decreases and converges to 0. Since $\mathcal{F}(0) = [0,1]$, we have:

$$\lim_{n \rightarrow \infty} x_n = 0 \text{ and } 0 \in \mathcal{F}(0)$$

Thus, 0 is a fixed point of the mapping $\mathcal{F}$.

Theorem 3.2 (Existence of Fixed Point in Symmetric G-Cone Metric Space)

Let (M, d_G) be a complete G -cone metric space over a real Banach space V , equipped with a normal cone $\mathcal{C} \subset V$. Consider a multivalued mapping $\mathcal{F}: M \rightarrow \mathcal{K}(M)$, where the collection of all nonempty compact subsets of M is represented by $\mathcal{K}(M)$. Assume that for all $u, v, w \in M$, the following inequality holds:

$$H_G(\mathcal{F}(u), \mathcal{F}(v), \mathcal{F}(w)) \leq \alpha \cdot d_G(u, v, w)$$

For some constant $\alpha \in [0,1]$, where $\leq$ denotes the cone-induced partial ordering and H_G is the G -Hausdorff metric. Then, there exists $u^* \in M$ such that $u^* \in \mathcal{F}(u^*)$.

Proof

Sequence Generation

Let $u_0 \in M$ be any initial point. Since $\mathcal{F}(u_0)$ is compact and nonempty, pick $u_1 \in \mathcal{F}(u_0)$. Proceed inductively: assume $u_n \in M$ is known. Then, choose $u_{n+1} \in \mathcal{F}(u_n)$ such that:

$$d_G(u_n, u_{n+1}, u_{n+1}) \leq H_G(\mathcal{F}(u_{n-1}), \mathcal{F}(u_n), \mathcal{F}(u_n))$$

Applying the contractive property of $\mathcal{F}$:

$$H_G(\mathcal{F}(u_{n-1}), \mathcal{F}(u_n), \mathcal{F}(u_n)) \leq \alpha \cdot d_G(u_{n-1}, u_n, u_n)$$

Thus:

$$d_G(u_n, u_{n+1}, u_{n+1}) \leq \alpha \cdot d_G(u_{n-1}, u_n, u_n) \text{ for all } n \geq 1$$

Distance Recurrence Relation

Define $\delta_n := d_G(u_n, u_{n+1}, u_{n+1}) \in \mathcal{C}$. Then, from (3.1), we get the recurrence:

$$\delta_n \leq \alpha \cdot \delta_{n-1}$$

By applying (3.2) recursively:

$$\delta_n \leq \alpha^n \cdot \delta_0 \quad \forall n \in \mathbb{N}$$

Since $\alpha \in [0,1)$ and $\delta_0 \in C$, it follows that $\delta_n \rightarrow \theta_V$, the zero element of V .

Showing $\{u_n\}$ is G-Cauchy

Let $\epsilon \in V$ with $\epsilon \gg \theta_V$. Since $\delta_n \rightarrow \theta_V$, there exists $N_0 \in \mathbb{N}$ such that:

$$\delta_n \ll \frac{1-\alpha}{2} \epsilon \text{ for all } n \geq N_0$$

Using the G-cone triangle inequality (property G 5), for $m > n$, we have:

$$d_G(u_n, u_m, u_m) \leq \sum_{k=n}^{m-1} d_G(u_k, u_{k+1}, u_{k+1}) = \sum_{k=n}^{m-1} \delta_k$$

From (3.3):

$$\sum_{k=n}^{m-1} \delta_k \leq \sum_{k=n}^{\infty} \alpha^k \cdot \delta_0 = \frac{\alpha^n}{1-\alpha} \cdot \delta_0$$

As $n \rightarrow \infty$, the upper bound $\frac{\alpha^n}{1-\alpha} \cdot \delta_0 \rightarrow \theta_V$, proving that $\{u_n\}$ is G-Cauchy.

Existence of the Limit

Because (M, d_G) is complete, the Cauchy sequence $\{u_n\}$ converges to some point $u^* \in M$, such that:

$$\lim_{n \rightarrow \infty} d_G(u_n, u^*, u^*) = \theta_V$$

Verification that $u^* \in \mathcal{F}(u^*)$

Using the triangle inequality again:

$$d_G(u_{n+1}, u^*, u^*) \leq d_G(u_{n+1}, u_n, u_n) + d_G(u_n, u^*, u^*)$$

Taking the limit as $n \rightarrow \infty$, and knowing that both terms on the right approach θ_V , we get:

$$\lim_{n \rightarrow \infty} d_G(u_{n+1}, u^*, u^*) = \theta_V$$

Hence, $u_{n+1} \rightarrow u^*$. Since each $u_{n+1} \in \mathcal{F}(u_n)$, and $\mathcal{F}$ has compact values, it follows by upper semicontinuity (or closed graph property) that:

$$u^* \in \mathcal{F}(u^*)$$

Thus, u^* is a fixed point of the mapping $\mathcal{F}$.

It has been demonstrated that Theorem 3.2 makes it simple to determine the fixed point result for multivalued mappings in symmetric G-cone metric spaces. The theorem generalizes fixed point theorems from the G-metric to the vector-valued case, by imposing generalized G-Hausdorff contraction condition on a cone induced partial order. We recursively define a sequence such that the observables $\{u_n\}$ are contained in the image of their successive term. The recurrence relation $\delta_n \leq \alpha^n \delta_0$ guarantees that the sequence verifies a contractive property in the cone. The sequence is G-Cauchy, using the triangle inequality for Gcone. The compactness of the space ensures convergence to a limit point $u^* \in M$. The upper semi continuity (or closed graph property) of the mapping rationalizes that $u^* \in \mathcal{F}(u^*)$. This theorem is not only an extension of scalar-valued fixed-point results but also a rigorous analytical framework for vector-valued multivalued mappings under cone structures.

Corollary 1 (Unique Fixed Point for Single-Valued Contractive Map in G-Cone Metric Space)

Statement: Let (M, d_G) be a complete symmetric G-cone metric space defined over a real Banach space V , equipped with a normal cone $C \subset V$. Assume that a single-valued mapping $\mathcal{T}: M \rightarrow M$ satisfies the contractive condition:

$$d_G(\mathcal{T}u, \mathcal{T}v, \mathcal{T}v) \leq \kappa \cdot d_G(u, v, v), \quad \forall u, v \in M$$

for some $\kappa \in [0,1)$. Then $\mathcal{T}$ has a unique fixed point $u^* \in M$ such that $\mathcal{T}(u^*) = u^*$.

Justification: By interpreting $\mathcal{T}$ as a degenerate multivalued mapping-i.e., defining $\mathcal{T}(x) = \{\mathcal{T}x\}$ the standard G-Hausdorff contraction condition simplifies to the inequality involving only the G-metric. Since the contractive assumption aligns with the framework of the multivalued fixed point theorem, and all conditions are satisfied, the existence and uniqueness of a fixed point follow immediately.

Corollary 2 (Fixed Point Existence in Compact G-Cone Metric Space)

Statement: Let (M, d_G) be a compact, symmetric G -cone metric space over a Banach space V , with a normal cone $C \subset V$. Suppose a multivalued mapping $\mathcal{F}: M \rightarrow \mathcal{K}(M)$ satisfies:

$$H_G(\mathcal{F}(u), \mathcal{F}(v), \mathcal{F}(w)) \leq \kappa \cdot d_G(u, v, w), \forall u, v, w \in M$$

for a constant $\kappa \in [0, 1)$. Then, there exists at least one point $u^* \in M$ such that $u^* \in \mathcal{F}(u^*)$.

Justification:

The compactness of M ensures completeness, and all conditions of the general fixed point theorem are satisfied. Hence, the existence of a fixed point is guaranteed.

Example (Illustration in Compact G-Cone Metric Structure)

Let $M = [0, 1]$ and define a G -metric $d_G: M \times M \times M \rightarrow \mathbb{R}_+$ by:

$$d_G(a, b, c) = |a - b| + |b - c| + |c - a|$$

Let the Banach space $V = \mathbb{R}$ and define the cone $C = [0, \infty)$, which forms a normal cone in $\mathbb{R}$.

Define a multivalued operator $\mathcal{F}: M \rightarrow \mathcal{K}(M)$ by:

$$\mathcal{F}(x) = \left[\frac{x}{3}, \frac{x+1}{3} \right]$$

This mapping assigns to each $x \in [0, 1]$ a compact interval within $[0, 1]$.

We claim that $\mathcal{F}$ satisfies the contraction condition:

$$H_G(\mathcal{F}(x), \mathcal{F}(y), \mathcal{F}(z)) \leq \frac{1}{3} \cdot d_G(x, y, z)$$

for all $x, y, z \in M$. The Hausdorff distance between such intervals behaves linearly with respect to their endpoints, and thus scales with the input G -distance. Since $\frac{1}{3} < 1$, the contraction condition is satisfied, and hence, by the earlier corollary, a fixed point exists.

Corollary 1 ensures uniqueness under contraction for single-valued maps by reducing the problem to a degenerate multivalued case. Corollary 2 extends existence results to compact domains, where the completeness is guaranteed by compactness. The example demonstrates practical applicability using a basic linear mapping over the unit interval, showing how fixed point theory can apply in concrete scenarios even in cone-ordered spaces.

4. Conclusion

In this article a common and generalized theoretical framework of fixed point theorems for multivalued contractive type mappings is given in G -metric and G -cone metric spaces. It is also shown that some previous main results due to Nadler, Mustafa-Sims and Huang-Zhang can be easily deduced from our results. Contrarily to scalar-oriented or set-theoretic methods, the one developed here provides convergent sequences based on recursive inequalities with stronger convergence properties for the both scalar and vector-valued multivalued mappings. One of the main elements of this work has consisted in showing how the G -cone metric structure contains the G -metric structure as a special case. This discussion allows more flexible analysis, in particular for Banach spaces which have a natural partial ordering in the sense of normal cones. Compactness, usc or contractive properties are such that the fixed-point existence follows, and they will facilitate proofs of convergence and adaptations of algorithms. Our results enrich not only the theory of fixed points but also its applications to various fields of mathematics and computer science. This encompasses operator theory, nonlinear analysis, optimisation, and systems described by multivalued dynamics. The template is a convenient tool in stability, equilibrium, and convergence analysis of these complicated systems.

For future studies, there are some promising directions. One possible generalization is to utilize these results for probabilistic, intuitionistic fuzzy, or fuzzy G -cone metric spaces, which can play an important role in modeling uncertainty real problems. Furthermore, the efficient algorithms with the constructed iterative sequences can be useful in practical applications of various problems in applied mathematics and engineering. Another interesting direction is that our approach may be generalized to the equilibrium problem, variational inclusion, and the game theory model with multivalued strategies. In addition, the combination of fractional derivative/integral-type brakes with this G -cone framework may result in powerful tools for modeling memory-dependent or hereditary systems. Finally, this paper provides with a solid framework of the theoretical and applied developments of the fixed-point theory in the abstract metric type of and cone-metric type of spaces.

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