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## Numerical solution of three-parameter eigenvalue problems using Newton's method

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### Abstract

Many mathematical problems in science and engineering can be expressed in terms of differential equations. When solving certain boundary value problems in partial differential equations using the separation-of-variables method, one frequently encounters multiparameter eigenvalue problems. A three-parameter eigenvalue problem is a specific type of such multiparameter problems. However, numerical research on eigenvalue problems involving three or more parameters remains limited, highlighting the need for further study. This paper focuses on computing numerical solutions to three-parameter eigenvalue problems using Newton's method. Numerical results are provided to demonstrate the effectiveness of the proposed approach.

**Keywords:** Multiparameter, three-parameter, eigenvalue, eigenvector, Newton's method, Gerschgorin theorem

### 1. Introduction

The multiparameter eigenvalue problem generalizes the classical one-parameter eigenvalue problem. A three-parameter eigenvalue problem is a specific case within this broader class. Such problems frequently arise when separation of variables is applied to boundary value problems associated with partial differential equations (Volkmer <sup>[14]</sup>). They naturally occur in many areas of mathematical physics. Examples include the dynamical analysis of a homogeneous beam under a vertical load (Collatz <sup>[4]</sup>), the vibrating membrane problem (Roach <sup>[12]</sup>), and the study of divergent flow between non-parallel planes in an electrically conducting fluid subjected to a transverse magnetic field (Srivastava and Hazarika <sup>[13]</sup>).

A three-parameter eigenvalue problem in matrix form is as follows

$$A_{10}x = \lambda_1 A_{11}x + \lambda_2 A_{12}x + \lambda_3 A_{13}x$$

$$A_{20}y = \lambda_1 A_{21}y + \lambda_2 A_{22}y + \lambda_3 A_{23}y \quad (1.1.1)$$

$$A_{30}z = \lambda_1 A_{31}z + \lambda_2 A_{32}z + \lambda_3 A_{33}z$$

where  $\lambda_i \in \mathbb{C}, i = 1, 2, 3$

$$x \in \mathbb{C}^n \setminus \{0\}; A_{10}, A_{11}, A_{12}, A_{13} \in \mathbb{C}^{n \times n}$$

$$y \in \mathbb{C}^m \setminus \{0\}; A_{20}, A_{21}, A_{22}, A_{23} \in \mathbb{C}^{m \times m}$$

$$z \in \mathbb{C}^p \setminus \{0\}; A_{30}, A_{31}, A_{32}, A_{33} \in \mathbb{C}^{p \times p}$$

The equation (1.1.1) can be reduced to a system of three one-parameter eigenvalue problems

$$\Delta_1 u = \lambda_1 \Delta_0 u \quad (1.1.2)$$

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$$\Delta_2 u = \lambda_2 \Delta_0 u \quad (1.1.3)$$

$$\Delta_3 u = \lambda_3 \Delta_0 u \quad (1.1.4)$$

where  $\Delta_0, \Delta_1, \Delta_2, \Delta_3$  are  $nmp \times nmp$  dimensional matrices defined as

$$\begin{aligned} \Delta_0 = & A_{11} \otimes A_{22} \otimes A_{33} - A_{11} \otimes A_{23} \otimes A_{32} + A_{12} \otimes A_{23} \otimes A_{31} \\ & - A_{12} \otimes A_{21} \otimes A_{33} + A_{13} \otimes A_{21} \otimes A_{32} - A_{13} \otimes A_{22} \otimes A_{31} \end{aligned} \quad (1.1.5)$$

$$\begin{aligned} \Delta_1 = & A_{10} \otimes A_{22} \otimes A_{33} - A_{10} \otimes A_{23} \otimes A_{32} + A_{12} \otimes A_{23} \otimes A_{30} \\ & - A_{12} \otimes A_{20} \otimes A_{33} + A_{13} \otimes A_{20} \otimes A_{32} - A_{13} \otimes A_{22} \otimes A_{30} \end{aligned} \quad (1.1.6)$$

$$\begin{aligned} \Delta_2 = & A_{11} \otimes A_{20} \otimes A_{33} - A_{11} \otimes A_{23} \otimes A_{30} + A_{10} \otimes A_{23} \otimes A_{31} \\ & - A_{10} \otimes A_{21} \otimes A_{33} + A_{13} \otimes A_{21} \otimes A_{30} - A_{13} \otimes A_{20} \otimes A_{31} \end{aligned} \quad (1.1.7)$$

$$\begin{aligned} \Delta_3 = & A_{11} \otimes A_{22} \otimes A_{30} - A_{11} \otimes A_{20} \otimes A_{32} + A_{12} \otimes A_{20} \otimes A_{31} \\ & - A_{12} \otimes A_{21} \otimes A_{30} + A_{10} \otimes A_{21} \otimes A_{32} - A_{10} \otimes A_{22} \otimes A_{31} \end{aligned} \quad (1.1.8)$$

and  $u = x \otimes y \otimes z$

In this paper we discuss Newton's method for three-parameter eigenvalue problems. We consider the problem (1.1.1) is non-singular in our entire discussion. To apply Newton's method for three-parameter problems some initial approximations are required. Here the Gerschgorin theorem is applied to find the initial approximation for Newton's Method.

Finally numerical examples are given to estimate the eigenvalues and eigenvectors of a three-parameter eigenvalue problems using Newton's Method based on the Gerschgorin theorem.

## 1.2 Newton's method for three-parameter eigenvalue problems

To apply Newton's method we consider the three-parameter eigenvalue problems (1.1.1) as

$$F(u) = 0$$

$$\text{where } F(u) = \begin{bmatrix} A_{10}x - \lambda_1 A_{11}x - \lambda_2 A_{12}x - \lambda_3 A_{13}x \\ A_{20}y - \lambda_1 A_{21}y - \lambda_2 A_{22}y - \lambda_3 A_{23}y \\ A_{30}z - \lambda_1 A_{31}z - \lambda_2 A_{32}z - \lambda_3 A_{33}z \\ -\frac{1}{2}x^T x + \frac{1}{2} \\ -\frac{1}{2}y^T y + \frac{1}{2} \\ -\frac{1}{2}z^T z + \frac{1}{2} \end{bmatrix} \quad (1.2.1)$$

$$\text{and } u = [x, y, z, \lambda_1, \lambda_2, \lambda_3]^T \in \mathbb{C}^{n+m+p+3}$$

In this paper we have considered three-parameter eigenvalue problems with real coefficients only. Under some relevant changes, the theory will be applicable in the complex field also.

To apply Newton's method for three-parameter eigenvalue problems  $F(u) = 0$ , the jacobian matrix  $F_u(u)$  must be non-singular. The following algorithm can be used for Newton's Iterative Method for three-parameter eigenvalue problems

1. Start with an initial approximations  $(x_0, y_0, z_0)$  to the eigenvectors where  $\|x_0\| = 1, \|y_0\| = 1, \|z_0\| = 1$  and  $(\lambda_1^0, \lambda_2^0, \lambda_3^0)$  to the eigenvalue  $(\lambda_1, \lambda_2, \lambda_3)$ .
2. Check determinant of matrices  $(A_{10} - \lambda_1^k A_{11} - \lambda_2^k A_{12} - \lambda_3^k A_{13}), (A_{20} - \lambda_1^k A_{21} - \lambda_2^k A_{22} - \lambda_3^k A_{23}), (A_{30} - \lambda_1^k A_{31} - \lambda_2^k A_{32} - \lambda_3^k A_{33})$  and if either are equal to 0, perturb  $(\lambda_1^k, \lambda_2^k, \lambda_3^k)$  slightly.
3. Solve the following equations

$$[A_{10} - \lambda_1^k A_{11} - \lambda_2^k A_{12} - \lambda_3^k A_{13}]u_k = A_{11}x_k$$

$$[A_{10} - \lambda_1^k A_{11} - \lambda_2^k A_{12} - \lambda_3^k A_{13}]v_k = A_{12}x_k$$

$$[A_{10} - \lambda_1^k A_{11} - \lambda_2^k A_{12} - \lambda_3^k A_{13}]w_k = A_{13}x_k$$

$$[A_{20} - \lambda_1^k A_{21} - \lambda_2^k A_{22} - \lambda_3^k A_{23}]p_k = A_{21}y_k$$

$$[A_{20} - \lambda_1^k A_{21} - \lambda_2^k A_{22} - \lambda_3^k A_{23}]q_k = A_{22}y_k$$

$$[A_{20} - \lambda_1^k A_{21} - \lambda_2^k A_{22} - \lambda_3^k A_{23}]r_k = A_{23}y_k$$

$$[A_{30} - \lambda_1^k A_{31} - \lambda_2^k A_{32} - \lambda_3^k A_{33}]a_k = A_{31}z_k$$

$$[A_{30} - \lambda_1^k A_{31} - \lambda_2^k A_{32} - \lambda_3^k A_{33}]b_k = A_{32}z_k$$

$$[A_{30} - \lambda_1^k A_{31} - \lambda_2^k A_{32} - \lambda_3^k A_{33}]c_k = A_{33}z_k$$

#### 4. Set up and solve the following system

$$\begin{pmatrix} x_k^T u_k & x_k^T v_k & x_k^T w_k \\ y_k^T p_k & y_k^T q_k & y_k^T r_k \\ z_k^T a_k & z_k^T b_k & z_k^T c_k \end{pmatrix} \begin{pmatrix} \Delta\lambda_1^k \\ \Delta\lambda_2^k \\ \Delta\lambda_3^k \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(x_k^T x_k + 1) \\ \frac{1}{2}(y_k^T y_k + 1) \\ \frac{1}{2}(z_k^T z_k + 1) \end{pmatrix}$$

#### 5. Update the approximate eigenvectors

$$x_{k+1} = \Delta\lambda_1^k u_k + \Delta\lambda_2^k v_k + \Delta\lambda_3^k w_k$$

$$y_{k+1} = \Delta\lambda_1^k p_k + \Delta\lambda_2^k q_k + \Delta\lambda_3^k r_k$$

$$z_{k+1} = \Delta\lambda_1^k a_k + \Delta\lambda_2^k b_k + \Delta\lambda_3^k c_k$$

#### 6. Normalise the approximated eigenvectors

$$x_{k+1} = \frac{x_{k+1}}{\|x_{k+1}\|}$$

$$y_{k+1} = \frac{y_{k+1}}{\|y_{k+1}\|}$$

$$z_{k+1} = \frac{z_{k+1}}{\|z_{k+1}\|}$$

### 1.3 Bounds of eigenvalues: Gerschgorin theorem

From section 1.2, it has been observed that to apply Newton's method for three-parameter eigenvalue problem we need some initial approximate eigenvalue and eigenvectors. To find the initial eigenvalue Gerschgorin theorem is used. Gerschgorin theorem is widely applicable to estimate the rough bounds of the eigenvalues of a matrix. According to Le Veque <sup>[111]</sup> the statement of the Gerschgorin theorem is as follows.

#### 1.3.1 Theorem

Let  $A \in C^{n \times n}$  and let  $D_i$  be the closed disk in the complex plane centered at  $a_{ii}$  with radius given by the row sum  $r_i = \sum_{j \neq i} |a_{ij}|$ ;  $D_i = \{z \in C \mid |z - a_{ii}| \leq r_i\}$ . Then all the eigenvalues of  $A$  lie in the union of the disks  $D_i$  for  $i = 1, 2, \dots, n$ .

If some set of  $k$  overlapping disks is disjoint from all the other disks, then exactly  $k$  eigenvalues lie in the union of these  $k$  disks.

#### 1.4 Numerical example

We now present a numerical example to show the behaviour and application of the Newton's method using Gerschgorin theorem. First of all, a three-parameter eigenvalue problem is considered. Then Gerschgorin theorem is used to find the initial approximation for Newton's method. Finally, Newton's method is used to solve the same problem.

Consider the three- parameter eigenvalue problem

$$\begin{aligned} \begin{pmatrix} 6 & 8 \\ 8 & 19 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \lambda_1 \begin{pmatrix} 16 & 18 \\ 18 & 39 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 26 & 28 \\ 28 & 59 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \lambda_3 \begin{pmatrix} 36 & 38 \\ 38 & 79 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ \begin{pmatrix} 9 & 7 \\ 7 & 11 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \lambda_1 \begin{pmatrix} 41 & 43 \\ 43 & 89 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 55 & 65 \\ 65 & 145 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \lambda_3 \begin{pmatrix} 69 & 67 \\ 67 & 131 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ \begin{pmatrix} 14 & 12 \\ 12 & 21 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &= \lambda_1 \begin{pmatrix} 126 & 78 \\ 78 & 84 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 316 & 198 \\ 198 & 219 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \\ &+ \lambda_3 \begin{pmatrix} 242 & 156 \\ 156 & 183 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \end{aligned} \tag{1.4.1}$$

**Table 1:** Estimate of  $(\lambda_1, \lambda_2, \lambda_3)$  and  $x, y, z$  for  $(\lambda_1^0, \lambda_2^0, \lambda_3^0) = (-0.01, -0.15, 0.2)$ ,  $x_0 = \begin{pmatrix} 1 \\ -0.05 \end{pmatrix}$ ,  $y_0 = \begin{pmatrix} 1 \\ -0.05 \end{pmatrix}$ ,  $z_0 = \begin{pmatrix} 1 \\ -0.05 \end{pmatrix}$ 

| Initial eigenvalue<br>$(\lambda_1^k, \lambda_2^k, \lambda_3^k)$ | Initial eigenvector                                                                                                                                                      | Iteration | Eigenvalue<br>$(\lambda_1^{k+1}, \lambda_2^{k+1}, \lambda_3^{k+1})$ | eigenvector                                                                                                                                                              | $\ \lambda^{k+1} - \lambda^k\ $ |
|-----------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------|---------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------|
| $(-0.01, -0.15, 0.2)$                                           | $x_0 = \begin{pmatrix} 1 \\ -0.05 \end{pmatrix}$<br>$y_0 = \begin{pmatrix} 1 \\ -0.05 \end{pmatrix}$<br>$z_0 = \begin{pmatrix} 1 \\ -0.05 \end{pmatrix}$                 | 1         | $(0.0444, -0.1563, 0.2295)$                                         | $x_1 = \begin{pmatrix} 0.9897 \\ -0.2419 \end{pmatrix}$<br>$y_1 = \begin{pmatrix} 1 \\ -0.0350 \end{pmatrix}$<br>$z_1 = \begin{pmatrix} 0.9907 \\ -0.2211 \end{pmatrix}$ | 0.0622                          |
| $(0.0444, -0.1563, 0.2295)$                                     | $x_1 = \begin{pmatrix} 0.9897 \\ -0.2419 \end{pmatrix}$<br>$y_1 = \begin{pmatrix} 1 \\ -0.0350 \end{pmatrix}$<br>$z_1 = \begin{pmatrix} 0.9907 \\ -0.2211 \end{pmatrix}$ | 2         | $(0.0036, -0.1505, 0.2483)$                                         | $x_2 = \begin{pmatrix} 0.9538 \\ -0.3158 \end{pmatrix}$<br>$y_2 = \begin{pmatrix} 1 \\ -0.0552 \end{pmatrix}$<br>$z_2 = \begin{pmatrix} 0.9575 \\ -0.3073 \end{pmatrix}$ | 0.0453                          |
| $(0.0036, -0.1505, 0.2483)$                                     | $x_2 = \begin{pmatrix} 0.9538 \\ -0.3158 \end{pmatrix}$<br>$y_2 = \begin{pmatrix} 1 \\ -0.0552 \end{pmatrix}$<br>$z_2 = \begin{pmatrix} 0.9575 \\ -0.3073 \end{pmatrix}$ | 3         | $(0.0000, -0.1500, 0.2500)$                                         | $x_3 = \begin{pmatrix} 0.9638 \\ -0.3212 \end{pmatrix}$<br>$y_3 = \begin{pmatrix} 1 \\ -0.0476 \end{pmatrix}$<br>$z_3 = \begin{pmatrix} 0.9639 \\ -0.3212 \end{pmatrix}$ | 0.0040                          |

**Table 2:** Estimate of  $(\lambda_1, \lambda_2, \lambda_3)$  and  $x, y, z$  for  $(\lambda_1^0, \lambda_2^0, \lambda_3^0) = (1.5, 0.34, 1.45)$ ,  $x_0 = \begin{pmatrix} 1.1 \\ 0.03 \end{pmatrix}$ ,  $y_0 = \begin{pmatrix} -0.05 \\ 1 \end{pmatrix}$ ,  $z_0 = \begin{pmatrix} 0.02 \\ 1.25 \end{pmatrix}$ 

| Initial eigenvalue<br>$(\lambda_1^k, \lambda_2^k, \lambda_3^k)$ | Initial eigenvector                                                                                                                                                      | Iteration | Eigenvalue<br>$(\lambda_1^{k+1}, \lambda_2^{k+1}, \lambda_3^{k+1})$ | eigenvector                                                                                                                                                              | $\ \lambda^{k+1} - \lambda^k\ $ |
|-----------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------|---------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------|
| $(1.5, 0.34, 1.45)$                                             | $x_0 = \begin{pmatrix} 1.1 \\ 0.03 \end{pmatrix}$<br>$y_0 = \begin{pmatrix} -0.05 \\ 1 \end{pmatrix}$<br>$z_0 = \begin{pmatrix} 0.02 \\ 1.25 \end{pmatrix}$              | 1         | $(-1.9323, 0.6257, 0.6695)$                                         | $x_1 = \begin{pmatrix} 0.9787 \\ 0.0744 \end{pmatrix}$<br>$y_1 = \begin{pmatrix} -0.0250 \\ 0.9759 \end{pmatrix}$<br>$z_1 = \begin{pmatrix} 0.0283 \\ 1 \end{pmatrix}$   | 3.5315                          |
| $(-1.9323, 0.6257, 0.6695)$                                     | $x_1 = \begin{pmatrix} 0.9787 \\ 0.0744 \end{pmatrix}$<br>$y_1 = \begin{pmatrix} -0.0250 \\ 0.9759 \end{pmatrix}$<br>$z_1 = \begin{pmatrix} 0.0283 \\ 1 \end{pmatrix}$   | 2         | $(-0.2870, -0.0453, 0.3280)$                                        | $x_2 = \begin{pmatrix} 1 \\ -0.4435 \end{pmatrix}$<br>$y_2 = \begin{pmatrix} -0.0733 \\ 0.9621 \end{pmatrix}$<br>$z_2 = \begin{pmatrix} -0.0515 \\ 0.9652 \end{pmatrix}$ | 1.8094                          |
| $(-0.2870, -0.0453, 0.3280)$                                    | $x_2 = \begin{pmatrix} 1 \\ -0.4435 \end{pmatrix}$<br>$y_2 = \begin{pmatrix} -0.0733 \\ 0.9621 \end{pmatrix}$<br>$z_2 = \begin{pmatrix} -0.0515 \\ 0.9652 \end{pmatrix}$ | 3         | $(-0.2560, 0.0076, 0.2455)$                                         | $x_3 = \begin{pmatrix} 0.6770 \\ -0.2045 \end{pmatrix}$<br>$y_3 = \begin{pmatrix} -1 \\ 0.6253 \end{pmatrix}$<br>$z_3 = \begin{pmatrix} -0.3564 \\ 0.6814 \end{pmatrix}$ | 0.1028                          |
| $(-0.2560, 0.0076, 0.2455)$                                     | $x_3 = \begin{pmatrix} 0.6770 \\ -0.2045 \end{pmatrix}$<br>$y_3 = \begin{pmatrix} -1 \\ 0.6253 \end{pmatrix}$<br>$z_3 = \begin{pmatrix} -0.3564 \\ 0.6814 \end{pmatrix}$ | 4         | $(-0.2484, -0.0032, 0.2523)$                                        | $x_4 = \begin{pmatrix} 1 \\ -0.3329 \end{pmatrix}$<br>$y_4 = \begin{pmatrix} -0.5623 \\ 1 \end{pmatrix}$<br>$z_4 = \begin{pmatrix} -0.4603 \\ 0.9191 \end{pmatrix}$      | 0.0149                          |
| $(-0.2484, -0.0032, 0.2523)$                                    | $x_4 = \begin{pmatrix} 1 \\ -0.3329 \end{pmatrix}$<br>$y_4 = \begin{pmatrix} -0.5623 \\ 1 \end{pmatrix}$<br>$z_4 = \begin{pmatrix} -0.4603 \\ 0.9191 \end{pmatrix}$      | 5         | $(-0.25, -0.0001, 0.2501)$                                          | $x_5 = \begin{pmatrix} 1 \\ -0.3333 \end{pmatrix}$<br>$y_5 = \begin{pmatrix} -0.4744 \\ 0.9523 \end{pmatrix}$<br>$z_5 = \begin{pmatrix} -0.4709 \\ 0.9419 \end{pmatrix}$ | 0.0041                          |
| $(-0.25, -0.0001, 0.2501)$                                      | $x_5 = \begin{pmatrix} 1 \\ -0.3333 \end{pmatrix}$<br>$y_5 = \begin{pmatrix} -0.4744 \\ 0.9523 \end{pmatrix}$<br>$z_5 = \begin{pmatrix} -0.4709 \\ 0.9419 \end{pmatrix}$ | 6         | $(-0.25, 0.00, 0.25)$                                               | $x_6 = \begin{pmatrix} 1 \\ -0.3333 \end{pmatrix}$<br>$y_6 = \begin{pmatrix} -0.4717 \\ 0.9433 \end{pmatrix}$<br>$z_6 = \begin{pmatrix} -0.4714 \\ 0.9428 \end{pmatrix}$ | 1.4142e-004                     |

**Table 3:** Estimate of  $(\lambda_1, \lambda_2, \lambda_3)$  and  $x, y, z$  for  $(\lambda_1^0, \lambda_2^0, \lambda_3^0) = (-2, -0.5, 1.8)$ ,  $x_0 = \begin{pmatrix} -0.01 \\ 1.02 \end{pmatrix}$ ,  $y_0 = \begin{pmatrix} 1 \\ -0.03 \end{pmatrix}$ ,  $z_0 = \begin{pmatrix} 1.02 \\ 0.001 \end{pmatrix}$ 

| Initial eigenvalue<br>$(\lambda_1^k, \lambda_2^k, \lambda_3^k)$ | Initial eigenvector                                                                                                                                                      | Iteration | Eigenvalue<br>$(\lambda_1^{k+1}, \lambda_2^{k+1}, \lambda_3^{k+1})$ | eigenvector                                                                                                                                                              | $\ \lambda^{k+1} - \lambda^k\ $ |
|-----------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------|---------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------|
| $(-2, -0.5, 1.8)$                                               | $x_0 = \begin{pmatrix} -0.01 \\ 1.02 \end{pmatrix}$<br>$y_0 = \begin{pmatrix} 1 \\ -0.03 \end{pmatrix}$<br>$z_0 = \begin{pmatrix} 1.02 \\ 0.001 \end{pmatrix}$           | 1         | $(-1.5754, -0.3031, 1.2845)$                                        | $x_1 = \begin{pmatrix} -0.2808 \\ 0.9971 \end{pmatrix}$<br>$y_1 = \begin{pmatrix} 0.9904 \\ -0.3230 \end{pmatrix}$<br>$z_1 = \begin{pmatrix} 1 \\ -0.1714 \end{pmatrix}$ | 0.6963                          |
| $(-1.5754, -0.3031, 1.2845)$                                    | $x_1 = \begin{pmatrix} -0.2808 \\ 0.9971 \end{pmatrix}$<br>$y_1 = \begin{pmatrix} 0.9904 \\ -0.3230 \end{pmatrix}$<br>$z_1 = \begin{pmatrix} 1 \\ -0.1714 \end{pmatrix}$ | 2         | $(-1.6638, -0.2759, 1.2918)$                                        | $x_2 = \begin{pmatrix} -0.4659 \\ 0.9520 \end{pmatrix}$<br>$y_2 = \begin{pmatrix} 0.9891 \\ -0.3304 \end{pmatrix}$<br>$z_2 = \begin{pmatrix} 1 \\ -0.3341 \end{pmatrix}$ | 0.0928                          |
| $(-1.6638, -0.2759, 1.2918)$                                    | $x_2 = \begin{pmatrix} -0.4659 \\ 0.9520 \end{pmatrix}$<br>$y_2 = \begin{pmatrix} 0.9891 \\ -0.3304 \end{pmatrix}$<br>$z_2 = \begin{pmatrix} 1 \\ -0.3341 \end{pmatrix}$ | 3         | $(-1.6384, -0.2729, 1.2741)$                                        | $x_3 = \begin{pmatrix} -0.4714 \\ 0.9432 \end{pmatrix}$<br>$y_3 = \begin{pmatrix} 0.9995 \\ -0.3332 \end{pmatrix}$<br>$z_3 = \begin{pmatrix} 1 \\ -0.3334 \end{pmatrix}$ | 0.0311                          |
| $(-1.6384, -0.2729, 1.2741)$                                    | $x_3 = \begin{pmatrix} -0.4714 \\ 0.9432 \end{pmatrix}$<br>$y_3 = \begin{pmatrix} 0.9995 \\ -0.3332 \end{pmatrix}$<br>$z_3 = \begin{pmatrix} 1 \\ -0.3334 \end{pmatrix}$ | 4         | $(-1.6365, -0.2727, 1.2728)$                                        | $x_4 = \begin{pmatrix} -0.4714 \\ 0.9428 \end{pmatrix}$<br>$y_4 = \begin{pmatrix} 1 \\ -0.3333 \end{pmatrix}$<br>$z_4 = \begin{pmatrix} 1 \\ -0.3333 \end{pmatrix}$      | 0.0023                          |
| $(-1.6365, -0.2727, 1.2728)$                                    | $x_4 = \begin{pmatrix} -0.4714 \\ 0.9428 \end{pmatrix}$<br>$y_4 = \begin{pmatrix} 1 \\ -0.3333 \end{pmatrix}$<br>$z_4 = \begin{pmatrix} 1 \\ -0.3333 \end{pmatrix}$      | 5         | $(-1.6364, -0.2727, 1.2727)$                                        | $x_5 = \begin{pmatrix} -0.4714 \\ 0.9428 \end{pmatrix}$<br>$y_5 = \begin{pmatrix} 1 \\ -0.3333 \end{pmatrix}$<br>$z_5 = \begin{pmatrix} 1 \\ -0.3333 \end{pmatrix}$      | 1.4142e-004                     |
| $(-1.6364, -0.2727, 1.2727)$                                    | $x_5 = \begin{pmatrix} -0.4714 \\ 0.9428 \end{pmatrix}$<br>$y_5 = \begin{pmatrix} 1 \\ -0.3333 \end{pmatrix}$<br>$z_5 = \begin{pmatrix} 1 \\ -0.3333 \end{pmatrix}$      | 6         | $(-1.6364, -0.2727, 1.2727)$                                        | $x_6 = \begin{pmatrix} -0.4714 \\ 0.9428 \end{pmatrix}$<br>$y_6 = \begin{pmatrix} 1 \\ -0.3333 \end{pmatrix}$<br>$z_6 = \begin{pmatrix} 1 \\ -0.3333 \end{pmatrix}$      | 0                               |

**Table 4:** Estimate of  $(\lambda_1, \lambda_2, \lambda_3)$  and  $x, y, z$  for  $(\lambda_1^0, \lambda_2^0, \lambda_3^0) = (-0.4, -0.1, 1)$ ,  $x_0 = \begin{pmatrix} 0.01 \\ 1 \end{pmatrix}$ ,  $y_0 = \begin{pmatrix} 0.002 \\ 1.25 \end{pmatrix}$ ,  $z_0 = \begin{pmatrix} 0.8 \\ -0.001 \end{pmatrix}$ 

| Initial eigenvalue<br>$(\lambda_1^k, \lambda_2^k, \lambda_3^k)$ | Initial eigenvector                                                                                                                                                      | Iteration | Eigenvalue<br>$(\lambda_1^{k+1}, \lambda_2^{k+1}, \lambda_3^{k+1})$ | eigenvector                                                                                                                                                              | $\ \lambda^{k+1} - \lambda^k\ $ |
|-----------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------|---------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------|
| $(-0.4, -0.1, 1)$                                               | $x_0 = \begin{pmatrix} 0.01 \\ 1 \end{pmatrix}$<br>$y_0 = \begin{pmatrix} 0.002 \\ 1.25 \end{pmatrix}$<br>$z_0 = \begin{pmatrix} 0.8 \\ -0.001 \end{pmatrix}$            | 1         | $(1.2541, -0.7544, 0.1953)$                                         | $x_1 = \begin{pmatrix} -0.0318 \\ 0.9758 \end{pmatrix}$<br>$y_1 = \begin{pmatrix} -0.1411 \\ 1 \end{pmatrix}$<br>$z_1 = \begin{pmatrix} 0.9998 \\ 0.0493 \end{pmatrix}$  | 1.9524                          |
| $(1.2541, -0.7544, 0.1953)$                                     | $x_1 = \begin{pmatrix} -0.0318 \\ 0.9758 \end{pmatrix}$<br>$y_1 = \begin{pmatrix} -0.1411 \\ 1 \end{pmatrix}$<br>$z_1 = \begin{pmatrix} 0.9998 \\ 0.0493 \end{pmatrix}$  | 2         | $(-0.1187, -0.3527, 0.5766)$                                        | $x_2 = \begin{pmatrix} -1 \\ 0.8052 \end{pmatrix}$<br>$y_2 = \begin{pmatrix} -0.2532 \\ 0.8097 \end{pmatrix}$<br>$z_2 = \begin{pmatrix} 0.8373 \\ 0.0174 \end{pmatrix}$  | 1.4803                          |
| $(-0.1187, -0.3527, 0.5766)$                                    | $x_2 = \begin{pmatrix} -1 \\ 0.8052 \end{pmatrix}$<br>$y_2 = \begin{pmatrix} -0.2532 \\ 0.8097 \end{pmatrix}$<br>$z_2 = \begin{pmatrix} 0.8373 \\ 0.0174 \end{pmatrix}$  | 3         | $(-0.1867, -0.3361, 0.5933)$                                        | $x_3 = \begin{pmatrix} -0.4985 \\ 0.9915 \end{pmatrix}$<br>$y_3 = \begin{pmatrix} -0.4381 \\ 0.9030 \end{pmatrix}$<br>$z_3 = \begin{pmatrix} 1 \\ -0.2397 \end{pmatrix}$ | 0.0720                          |
| $(-0.1867, -0.3361, 0.5933)$                                    | $x_3 = \begin{pmatrix} -0.4985 \\ 0.9915 \end{pmatrix}$<br>$y_3 = \begin{pmatrix} -0.4381 \\ 0.9030 \end{pmatrix}$<br>$z_3 = \begin{pmatrix} 1 \\ -0.2397 \end{pmatrix}$ | 4         | $(-0.1854, -0.3398, 0.5974)$                                        | $x_4 = \begin{pmatrix} -0.4717 \\ 0.9435 \end{pmatrix}$<br>$y_4 = \begin{pmatrix} -0.4695 \\ 0.9383 \end{pmatrix}$<br>$z_4 = \begin{pmatrix} 1 \\ -0.3307 \end{pmatrix}$ | 0.0057                          |
| $(-0.1854, -0.3398, 0.5974)$                                    | $x_4 = \begin{pmatrix} -0.4717 \\ 0.9435 \end{pmatrix}$<br>$y_4 = \begin{pmatrix} -0.4695 \\ 0.9383 \end{pmatrix}$<br>$z_4 = \begin{pmatrix} 1 \\ -0.3307 \end{pmatrix}$ | 5         | $(-0.1856, -0.3402, 0.5979)$                                        | $x_5 = \begin{pmatrix} -0.4714 \\ 0.9429 \end{pmatrix}$<br>$y_5 = \begin{pmatrix} -0.4713 \\ 0.9426 \end{pmatrix}$<br>$z_5 = \begin{pmatrix} 1 \\ -0.3333 \end{pmatrix}$ | 6.7082e-004                     |
| $(-0.1856, -0.3402, 0.5979)$                                    | $x_5 = \begin{pmatrix} -0.4714 \\ 0.9429 \end{pmatrix}$<br>$y_5 = \begin{pmatrix} -0.4713 \\ 0.9426 \end{pmatrix}$<br>$z_5 = \begin{pmatrix} 1 \\ -0.3333 \end{pmatrix}$ | 6         | $(-0.1856, -0.3402, 0.5979)$                                        | $x_6 = \begin{pmatrix} -0.4714 \\ 0.9428 \end{pmatrix}$<br>$y_6 = \begin{pmatrix} -0.4714 \\ 0.9428 \end{pmatrix}$<br>$z_6 = \begin{pmatrix} 1 \\ -0.3333 \end{pmatrix}$ | 0                               |

**Table 5:** Estimate of  $(\lambda_1, \lambda_2, \lambda_3)$  and  $x, y, z$  for  $(\lambda_1^0, \lambda_2^0, \lambda_3^0) = (-0.23, -0.32, 0.6)$ ,  $x_0 = \begin{pmatrix} 0.01 \\ 1.1 \end{pmatrix}$ ,  $y_0 = \begin{pmatrix} 0.03 \\ 1.3 \end{pmatrix}$ ,  $z_0 = \begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix}$ 

| Initial eigenvalue $(\lambda_1^k, \lambda_2^k, \lambda_3^k)$ | Initial eigenvector                                                                                                                                                      | Iteration | Eigenvalue $(\lambda_1^{k+1}, \lambda_2^{k+1}, \lambda_3^{k+1})$ | eigenvector                                                                                                                                                              | $\ \lambda^{k+1} - \lambda^k\ $ |
|--------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------|------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------|
| $(-0.23, -0.32, 0.6)$                                        | $x_0 = \begin{pmatrix} 0.01 \\ 1.1 \end{pmatrix}$<br>$y_0 = \begin{pmatrix} 0.03 \\ 1.3 \end{pmatrix}$<br>$z_0 = \begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix}$               | 1         | $(-0.2456, -0.3131, 0.6064)$                                     | $x_1 = \begin{pmatrix} -0.3394 \\ 0.6530 \end{pmatrix}$<br>$y_1 = \begin{pmatrix} -0.3498 \\ 0.6776 \end{pmatrix}$<br>$z_1 = \begin{pmatrix} -1 \\ 0.9292 \end{pmatrix}$ | 0.0182                          |
| $(-0.2456, -0.3131, 0.6064)$                                 | $x_1 = \begin{pmatrix} -0.3394 \\ 0.6530 \end{pmatrix}$<br>$y_1 = \begin{pmatrix} -0.3498 \\ 0.6776 \end{pmatrix}$<br>$z_1 = \begin{pmatrix} -1 \\ 0.9292 \end{pmatrix}$ | 2         | $(-0.2335, -0.3170, 0.6051)$                                     | $x_2 = \begin{pmatrix} -0.4694 \\ 0.9390 \end{pmatrix}$<br>$y_2 = \begin{pmatrix} -0.4646 \\ 0.9298 \end{pmatrix}$<br>$z_2 = \begin{pmatrix} -0.5057 \\ 1 \end{pmatrix}$ | 0.0128                          |
| $(-0.2335, -0.3170, 0.6051)$                                 | $x_2 = \begin{pmatrix} -0.4694 \\ 0.9390 \end{pmatrix}$<br>$y_2 = \begin{pmatrix} -0.4646 \\ 0.9298 \end{pmatrix}$<br>$z_2 = \begin{pmatrix} -0.5057 \\ 1 \end{pmatrix}$ | 3         | $(-0.2367, -0.3159, 0.6053)$                                     | $x_3 = \begin{pmatrix} -0.4974 \\ 0.9947 \end{pmatrix}$<br>$y_3 = \begin{pmatrix} -0.4971 \\ 0.9943 \end{pmatrix}$<br>$z_3 = \begin{pmatrix} -0.5 \\ 1 \end{pmatrix}$    | 0.0034                          |
| $(-0.2367, -0.3159, 0.6053)$                                 | $x_3 = \begin{pmatrix} -0.4974 \\ 0.9947 \end{pmatrix}$<br>$y_3 = \begin{pmatrix} -0.4971 \\ 0.9943 \end{pmatrix}$<br>$z_3 = \begin{pmatrix} -0.5 \\ 1 \end{pmatrix}$    | 4         | $(-0.2368, -0.3158, 0.6053)$                                     | $x_4 = \begin{pmatrix} -0.4997 \\ 0.9994 \end{pmatrix}$<br>$y_4 = \begin{pmatrix} -0.4999 \\ 0.9994 \end{pmatrix}$<br>$z_4 = \begin{pmatrix} -0.5 \\ 1 \end{pmatrix}$    | 1.4142e-004                     |
| $(-0.2368, -0.3158, 0.6053)$                                 | $x_4 = \begin{pmatrix} -0.4997 \\ 0.9994 \end{pmatrix}$<br>$y_4 = \begin{pmatrix} -0.4999 \\ 0.9994 \end{pmatrix}$<br>$z_4 = \begin{pmatrix} -0.5 \\ 1 \end{pmatrix}$    | 5         | $(-0.2368, -0.3158, 0.6053)$                                     | $x_5 = \begin{pmatrix} -0.5 \\ 1 \end{pmatrix}$<br>$y_5 = \begin{pmatrix} -0.5 \\ 1 \end{pmatrix}$<br>$z_5 = \begin{pmatrix} -0.5 \\ 1 \end{pmatrix}$                    | 0                               |

From the above five tables the eigenvalues obtained by Newton's method for three-parameter eigenvalue problems, taking five different starting eigenvalues and eigenvectors are

$(0.0000, -0.1500, .2500)$ ,  $(-0.25, 0.00, 0.25)$ ,  $(-1.6364, -0.2727, 1.2727)$

$(-0.1856, -0.3402, 0.5979)$ ,  $(-0.2368, -0.3158, 0.6053)$

## 5 Conclusion

From the above five Tables it is observed that the successive differences between the eigenvalues are gradually decreased and it indicates that the method converges to exact solution. It is also observed that different eigenvalues are obtained by taking different starting values. From the above tables it is observed that convergence is slower at the beginning but increases when the approximations get closer to the exact eigenvalue. So when the initial approximation is close to the desired eigenvalues then the number of iteration is less. Therefore choosing a good initial approximation is very important for Newton's method. So in the present Chapter Gerschgorin theorem is applied on reduced system of three one-parameter eigenvalue problems (1.1.2), (1.1.2) and (1.1.3) separately to find the bounds of the eigenvalues. The main advantage of this method is that it gives both eigenvalues and eigenvectors at the same time.

## References

1. Atkinson FV. Multiparameter eigenvalue problems: matrices and compact operators. New York, London: Academic Press; 1972.
2. Bellman R. Introduction to matrix analysis. 2nd ed. Philadelphia: SIAM; 1997.
3. Burden RL, Faires JD. Numerical analysis. 6th ed. Pacific Grove (USA): Cole Publishing Company; 1997.
4. Collatz L. Multiparametric eigenvalue problems in inner-product spaces. Journal of Computer and System Sciences. 1968;2:333-341.
5. Hochstenbach ME, Plestenjak B. Backward error, condition numbers, and pseudospectrum for the multiparameter eigenvalue problem. Linear Algebra Appl. 2003;375:63-81.
6. Holmquist B. The direct product permuting matrices. Linear and Multilinear Algebra. 1985;17(2):117-141.
7. Horn RA, Johnson CR. Topics in matrix analysis. Cambridge: Cambridge University Press; 1991.
8. Konwar J. Certain studies of two-parameter eigenvalue problems [PhD thesis]. Assam: Dibrugarh University; 2002.
9. LeVeque RJ. Finite difference methods for ordinary and partial differential equations: steady-state and time-dependent problems. Philadelphia: SIAM; 1955.
10. Magnus JR, Neudecker H. Matrix differential calculus with applications in statistics and economics. Revised ed. Wiley Series in Probability and Statistics. Chichester: Wiley; 1999.
11. Marquis D. Gerschgorin's circle theorem for estimating the eigenvalues of matrix with known error bounds [Internet]. 2016. Available from: <http://math.stmarys-ca.edu/wp-content/uploads/2017/07/David-Marquis.pdf>
12. Roach GF. A Fredholm theory for multiparameter problems. Nieuw Arch Wisk. 1976;24:49-76.
13. Srivastava AC, Hazarika GC. Shooting method for the flow between non-parallel plates with magnetic field. Mathematical Forum. 1992;9(1):1-12.
14. Volkmer H. Multiparameter problems and expansion theorems. Lecture Notes in Mathematics 1356. New York: Springer-Verlag; 1988.