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## Construction of triangular design by an unreduced balanced incomplete block design

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### Abstract

Another important and highly investigated category of Partially Balanced Incomplete Block Designs (PBIBDs) with two associate classes is triangular designs. This paper describes a construction procedure of triangular designs with an unreduced Balanced Incomplete Block Design (BIBD). Triangular designs form a special case of PBIBDs, and are therefore also applicable to experimental design, in coding theory, and in finite geometry by their clear associate structure and incidence properties. The suggested construction offers a systematic structure through the parameters and incidence relationships of an unreduced BIBD to generate a triangular scheme of association. The relation between the parameters of the unreduced BIBD and those of the triangular design obtained is determined, and the conditions that are required to be the case are checked. A number of examples are provided to prove the usefulness of the proposed approach. This method underlines a combinatorial association between BIBDs and PBIBDs.

**Keywords:** Triangular design, Balanced Incomplete Block Design (BIBD), Partially Balanced Incomplete Block Design (PBIBD), unreduced balanced incomplete block design

### 1. Introduction

Combinatorial design theory is a well-established and active area of Discrete Mathematics, primarily concerned with the systematic arrangement of elements into subsets that satisfy specified balance and symmetry properties. Among the various classes of combinatorial design, BIBDs occupy a central position due to their rich theoretical structure and wide range of applications in Statistics, Coding theory, Experimental design, and Cryptography.

Triangular designs form an important subclass of block designs, closely related to triangular graphs and association schemes. These designs exhibit strong regularity conditions and play a significant role in the study of symmetric structures and algebraic combinatorics.

In this work, we explore a constructive approach to triangular designs using unreduced BIBDs. By utilizing block and incidence structure of an unreduced BIBD, we demonstrate how triangular designs can be systematically derived while preserving essential combinatorial properties. This paper's goal is to provide a technique for constructing triangular designs from unreduced BIBDs and to establish the corresponding design parameters. This approach not only generalizes existing construction techniques but also provides new insights into the relationship between BIBDs and triangular designs.

#### 1.1 Balanced incomplete block design

BIBDs are of central importance because of their balanced incidence structure. A BIBD is defined as an arrangement of  $v$  treatments into  $b$  blocks, each block containing  $k$  treatments ( $k < v$ ), such that every treatment appears in exactly  $r$  blocks or “every pair of distinct treatments occurs together in exactly  $\lambda$  blocks. The parameters of a BIBD have been denoted by  $(v, b, r, k, \lambda)$ .

**These parameters satisfy the standard relationships**

$$bk = vr, r(k - 1) = \lambda(v - 1), v \leq b.$$

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## 1.2 Partially balanced incomplete block design

A PBIBD with two associate classes consists of  $b$  blocks and  $v$  treatments arranged such that the following conditions are satisfied.

- Each of the  $v$  treatments appears exactly  $r$  times across the  $b$  blocks, where each block contains  $k$  treatments ( $k < v$ ), and no treatment is repeated within same block.
- Every pair of treatments satisfies the association scheme described below:
  - Any 2 treatments are classified either as 1<sup>st</sup> associates or as 2<sup>nd</sup> associates.
  - Each treatment has exactly  $n_i$  associates of the  $i^{\text{th}}$  type, where  $i = 1, 2$ .
  - For any two treatments that are  $i^{\text{th}}$  associates, the number of treatments that are simultaneously the  $j^{\text{th}}$  associate of the first treatment and the  $k^{\text{th}}$  associate of the second treatment is denoted by  $p_{jk}^i$ , and this number is independent of the particular pair chosen. Moreover.

$$p_{jk}^i = p_{kj}^i, i, j, k = 1, 2.$$

- Any pair of treatments that are  $i^{\text{th}}$  associates occur together in exactly  $\lambda_i$  blocks, for  $i = 1, 2$ .
- The quantities

$$v, b, r, k, \lambda_1, \lambda_2, \dots, \lambda_m, n_1, n_2, \dots, n_m$$

Are referred to as the parameters of the first kind, while the numbers are known as the parameters of the second kind.

$$p_{jk}^i (i, j, k = 1, 2, \dots, m)$$

It is known that these parameters satisfy certain standard relationships.

The parameters of the system satisfy several fundamental relationships that ensure internal consistency.

First, the total number of incidents can be expressed in two equivalent ways. On one hand, each of the  $b$  blocks contains  $k$  elements, while on the other hand, each of the  $v$  elements occurs in  $r$  blocks. Equating these two counts yields.

$$bk = vr$$

Let the system be partitioned into  $m$  distinct classes, where  $n_i$  denotes number of elements in  $i$ -th class. Since the total number of elements excluding a reference element is  $v - 1$ , the class sizes satisfy.

$$\sum_{i=1}^m n_i = v - 1$$

If  $\lambda_i$  represents number of times an element from the  $i$ -th class occurs together with the reference element, then summing over all classes gives the total number of such pairwise occurrences. This total must equal the number of blocks that contain reference element multiplied by the number of remaining elements in each block, leading to.

$$\sum_{i=1}^m n_i \lambda_i = r(k - 1)$$

Let  $p_{jk}^i$  denote the number of elements from class  $i$  that are associated with an ordered pair of classes  $(j, k)$ . Summing over all possible  $k$  gives.

$$\sum_{k=1}^m p_{jk}^i = \begin{cases} n_j, & \text{if } i \neq j \\ n_j - 1, & \text{if } i = j \end{cases}$$

Finally, symmetry in the distribution of elements across classes implies proportional consistency between the class sizes and the associated parameters, which can be written as.

$$n_i p_{jk}^i = n_j p_{ik}^j = n_k p_{ij}^k$$

If  $m = 2$ , then clearly

**The following relations hold**

$$n_1 + n_2 = v - 1$$

$$n_1 \lambda_1 + n_2 \lambda_2 = r(k - 1)$$

$$p_{11}^2 + p_{12}^2 = p_{11}^1 + p_{12}^1 + 1 = n_1$$

$$p_{21}^1 + p_{22}^1 = p_{21}^2 + p_{22}^2 + 1 = n_2$$

$$n_1 p_{12}^1 = n_2 p_{11}^2 \text{ and } n_1 p_{22}^1 = n_2 p_{12}^2$$

Eight 1<sup>st</sup> kind factors determine the designs of 2 associate classes  $v, b, r, k, \lambda_1, \lambda_2, n_1, n_2$  and 6 parameters of 2<sup>nd</sup> kind  $p_{jk}^i = p_{kj}^i (i, j, k = 1, 2)$ . Parameters of 2<sup>nd</sup> kind might be demonstrated as elements of 2 symmetric matrices.

$$p_{jk}^1 = \begin{bmatrix} p_{11}^1 & p_{12}^1 \\ p_{21}^1 & p_{22}^1 \end{bmatrix}, p_{jk}^2 = \begin{bmatrix} p_{11}^2 & p_{12}^2 \\ p_{21}^2 & p_{22}^2 \end{bmatrix}$$

## 1.3 Triangular design

A triangular design belongs to the class of PBIBDs, in which the association scheme is established through a triangular relationship defined on unordered pairs of elements.

$$v = \frac{n(n-1)}{2}$$

Let Treatments be placed within an  $n \times n$  array such that all positions along the principal diagonal are left vacant. The remaining  $\frac{n(n-1)}{2}$  treatments are distributed symmetrically in the cells located either above or below the principal diagonal. This symmetric placement ensures that the overall configuration remains invariant with respect to the principal diagonal.

Under this arrangement, any two treatments occurring within same row or same column of the array are said to be associated, and such pairs are classified as second associates.

The numbers of first and second associates are given by:

$$n_1 = 2(n-2), n_2 = \frac{(n-2)(n-3)}{2}$$

The intersection matrices corresponding to the first and second associates are:

$$P_1 = \begin{pmatrix} p_{11}^1 & p_{12}^1 \\ p_{21}^1 & p_{22}^1 \end{pmatrix} = \begin{pmatrix} n-2 & \frac{n-3}{2} \\ n-3 & \frac{(n-3)(n-4)}{2} \end{pmatrix}$$

$$P_2 = \begin{pmatrix} p_{11}^2 & p_{12}^2 \\ p_{21}^2 & p_{22}^2 \end{pmatrix} = \begin{pmatrix} 4 & \frac{2n-8}{2} \\ 2n-8 & \frac{(n-4)(n-5)}{2} \end{pmatrix}$$

A triangular design is therefore a PBIBD constructed using a triangular configuration.

$$v = \frac{n(n-1)}{2}, b, r, k, \lambda_1, \lambda_2, n_1, n_2$$

The quantities are referred to as the parameters of the triangular design, and they satisfy the standard relationships

$$bk = vr$$

And

$$n_1\lambda_1 + n_2\lambda_2 = 2(n-2)\lambda_1 + \frac{(n-2)(n-3)}{2}\lambda_2 = r(k-1)$$

### 1.4 Unreduced balanced incomplete block design

An unreduced BIBD preserves all combinatorial relations, making it a useful foundation for constructing more complex designs. An unreduced BIBD with  $v$  treatments, in  $b$  blocks each of size  $k$  units, is created by taking every possible combination of  $k$  symbols ( $k \leq v$ ) in each of the blocks, to develop  $b = \binom{v}{k}$  blocks, for any fixed  $k$ .

An unreduced BIBD may be viewed as a universal collection of BIBDs defined for fixed parameters  $v$  and  $k$ . In this construction, the design consists of all possible subsets of size  $k$  chosen from a set of  $v$  elements, where each such subset constitutes a block. Consequently, this design includes every BIBD having the same values of  $v$  and  $k$ .

The associated parameters of this BIBD are given by

$$v^* = v, b^* = \binom{v}{k}, r^* = \binom{v-1}{k-1}, k^* = k, \lambda^* = \binom{v-2}{k-2}.$$

### 2. Construction of triangular design

**2.1 Theorem:** An unreduced BIBD with parameters  $v^* = v, b^* = \binom{v}{k}, r^* = \binom{v-1}{k-1}, k^* = k, \lambda^* = \binom{v-2}{k-2}$  [for  $v = \frac{n(n-1)}{2}, (n = 4, 5, 6, \dots)$  and  $k = v - 1$ ] will provide a 2 associate class Triangular PBIBD with the following parameters under the Triangular association scheme.

- $v^{**} = v, b^{**} = n, r^{**} = 2, k^{**} = n - 1, \lambda_1 = 1, \lambda_2 = 0$
- $v^{**} = \frac{(n-1)(n-2)}{2}, b^{**} = n - 1, r^{**} = 2, k^{**} = n - 2, \lambda_1 = 1, \lambda_2 = 0$

Proof,

Triangular PBIBD satisfies relations:

$$bk = vr$$

$$n_1\lambda_1 + n_2\lambda_2 = r(k-1)$$

We need to prove that a Triangular PBIBD satisfies conditions for existence of a PBIBD. First, we start with the 1<sup>st</sup> condition:

$$I. b^*k^* = n(n-1)$$

$$= \frac{n(n-1)}{2}(2)$$

$$= v^*r^*$$

Hence, the first condition of PBIBD is satisfied. We now proceed to verify the second condition required for the existence of a PBIBD.

$$n_1\lambda_1 + n_2\lambda_2 = 2(n-2) \cdot 1 + \frac{(n-2)(n-3)}{2} \cdot 0$$

$$= 2[(n-1) - 1]$$

$$= r^*(k^* - 1)$$

Thus, the second condition of PBIBD is also satisfied. Now, considering the parameters,

$$b^*k^* = (n-1)(n-2)$$

$$= \frac{(n-1)(n-2)}{2} \cdot 2$$

$$= v^*r^*$$

Hence, the first condition of PBIBD holds. Next, we verify the second condition.

$$n_1\lambda_1 + n_2\lambda_2 = 2(n-3) \cdot 1 + \frac{(n-3)(n-4)}{2} \cdot 0$$

$$= 2[(n-2) - 1]$$

$$= r^*(k^* - 1)$$

Therefore, the second condition of PBIBD is again satisfied. Consequently, a triangular PBIBD is obtained with parameters

$$v^* = v, b^* = n, r^* = 2, k^* = n - 1, \lambda_1 = 1, \lambda_2 = 0.$$

&

$$v^* = \frac{(n-1)(n-2)}{2}, b^* = n - 1, r^* = 2, k^* = n - 2, \lambda_1 = 1, \lambda_2 = 0$$

### 2.2 Example

Consider an unreduced BIBD having the parameters

$$v^* = 6, b^* = 6, r^* = 5, k^* = 5, \lambda^* = 4 [n = 4].$$

The corresponding blocks are given as:

Block 1 = (1,2,3,4,5),

Block 2 = (1,2,3,4,6),

Block 3 = (1,2,3,5,6),

Block 4 = (1,2,4,5,6),

Block 5 = (1,3,4,5,6),

Block 6 = (2,3,4,5,6).

From this configuration, a triangular association scheme is obtained, as shown below.

|   |   |   |   |
|---|---|---|---|
| * | 1 | 2 | 3 |
| 1 | * | 4 | 5 |
| 2 | 4 | * | 6 |
| 3 | 5 | 6 | * |

The resulting blocks of the triangular design are

Block 1 = (1,2,3),  
 Block 2 = (1,4,5),  
 Block 3 = (2,4,6),  
 Block 4 = (3,5,6).

Thus, the above construction yields a triangular design with parameters

$$v^{**} = 6, b^{**} = 4, r^{**} = 2, k^{**} = 3, \lambda_1 = 1, \lambda_2 = 0.$$

Verification

$$b^{**}k^{**} = 4 \times 3 = 12 = 6 \times 2 = v^{**}r^{**}$$

$$n_1\lambda_1 + n_2\lambda_2 = 4 \times 1 + 1 \times 0 = 4 = 2(3 - 1) = r^{**}(k^{**} - 1)$$

### 3. Conclusion

Suppose an unreduced BIBD exists with parameters

$$v^* = v, b^* = \binom{v}{k}, r^* = \binom{v-1}{k-1}, k^* = k, \lambda^* = \binom{v-2}{k-2},$$

Where,

$$v = \frac{n(n-1)}{2}, (n = 4, 5, 6, \dots) \text{ and } k = v - 1.$$

Under these conditions, a triangular design can be constructed.

The resulting triangular designs have the following parameter sets.

$$v^{**} = v, b^{**} = n, r^{**} = 2, k^{**} = n - 1, \lambda_1 = 1, \lambda_2 = 0$$

$$v^{**} = \frac{(n-1)(n-2)}{2}, b^{**} = n - 1, r^{**} = 2, k^{**} = n - 2, \lambda_1 = 1, \lambda_2 = 0$$

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