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T-Ordering in ternary semiring

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Abstract

We investigate the concept of minus ordering in Ternary Semiring. We derive some equivalent conditions for minus ordering by using generalized inverses and prove that T-ordering is a partial ordering. 2020 mathematical science classification: 16Y60, 16U90.

Keywords: Ternary semiring, regular, ordering, g-inverses

1. Introduction

In 1932, Lehmer ^[1] explored ternary algebraic system theory. He studied “triplexes”, which are ternary algebraic systems that finally turn out to be ternary groups. A generalization of ternary rings is called ternary semirings, as presented by Dutta *et al.* ^[2]. Ternary semiring is a superset of semiring. Negative integers form a ternary semiring but it does not form a semiring. In 1920, E.H. Moore established the concept of the generalized inverse of a matrix, defining the unique generalized inverse through the use of matrices projectors. “Penrose demonstrated in 1955 that Moore’s inverse is the unique matrix that can satisfy four equations ^[3]. The notion of regular element was first introduced by Von Neumann in 1936 ^[4]. Jian Miao Cen ^[5] introduced the concept of t-ordering in fuzzy matrices and he has discussed the relationships between the T-ordering and the generalized inverses.

In this paper we studied the concept of T-ordering for elements in ternary semiring as a generalization of the T-ordering for regular fuzzy matrices.

2. Preliminaries

In this work, T represents the Ternary semiring; it is abbreviated as T.S.R. throughout this paper.

Definition 2.1

The term “ternary semiring” refers to $T \neq \emptyset$, a binary addition, and a ternary multiplication called juxtaposition if $(T, +)$ is a semigroup that commutes that fulfilling the requirements listed below.

- $(abc)de = a(bcd)e = ab(cde)$,
- $(a + b)cd = acd + bcd$,
- $a(b + c)d = abd + acd$,
- $ab(c + d) = abc + abd$, for all a, b, d, e in T

Remark z^- is an natural example of a ternary semiring, which has ternary multiplication and regular addition.

- **Definition 2.2:** If an element a exists in T such that $ababa = a$, then a is considered regular. The element b is represented as a^- and is also known as the generalized inverse or 1-inverse of the element ' a ' and the set of all 1-inverses is denoted by $a\{1\}$. If every element in a T.S.R is regular, then we say T.S.R is regular.
- **Definition 2.3:** If there is an element a in $T \exists babab = b$, then a is considered anti-regular. The name for element b is 2-inverse of a and set of all 2-inverses of a is denoted by $a\{2\}$.

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The both conditions in Definition (2.2) and Definition (2.3) satisfied by a single element b is called 1-2 inverse and is denoted by $\hat{a}$. In Definition (2.2), if the ' b ' is unit then ' a ' is called unit regular.

Definition 2.4: The Moore-Penrose inverse of an element $a \in T$ if $\exists b \in T$ such that the following equations hold:

$$ababa = a \quad (2) \quad babab = b \quad (3) \quad (abe)^T = abe \quad (4) \quad (eba)^T = eba.$$

Where e is identity element, here b is unique and denoted by a^+

In Definition (2.4), the element b is satisfy the equations (1) and (3), then we say that element is $\{1,3\}$ -inverse of ' a ' and b satisfy the equation (1) and (4) then we say that b is $\{1,4\}$ -inverse of ' a '.

The transpose of a is written as a^T and the following properties of the transpose will be used:

$$a^{TT} = a,$$

$$(a + b)^T = a^T + b^T,$$

$$(abc)^T = c^T b^T a^T.$$

3. T-ordering in ternary semiring

Definition 4.1: The T-ordering $a \leq^T b$ in T is defined as $a \leq^T b$ iff $aa^T a = aa^T b = ba^T a$; $aaa^T = aba^T = baa^T$; $a^T aa = a^T ba = a^T ab$.

Example 1: Let $T = \{0,1,2,3,4\}$ be a ternary semiring with respect to modulo addition and modulo multiplication 5, the T-ordering defined as follows:

Let $a = 3$; $a^T = 2$; $b = 3$ satisfy $aa^T a = aa^T b = ba^T a$; ; $aaa^T = baa^T = aba^T$; $a^T aa = a^T ba = a^T ab$

$\cdot_5$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Throughout in this paper, let a^+ be Moore-Penrose inverse

Theorem 3.1: ^[8] Let $a, b \in T$ and a^+ exists. The subsequent items are therefore equal.

- $a \leq^T b$
- $aa^+ a = aa^+ b = ba^+ a$; $a^+ aa = a^+ ba = a^+ ab$; $aaa^+ = aba^+ = baa^+$;
- $a = aa^+ aa^+ b = ba^+ aa^+ a = aa^+ ba^+ a = baa^+ ba^+ ab$

Proof: (i) $\Rightarrow$

(ii) Since $a \leq^T b \Leftrightarrow aa^T a = aa^T b = ba^T a$; $a^T aa = a^T ba = a^T ab$; $aaa^T = aba^T = baa^T$

$$aa^+ a = a(a^+ aa^+ aa^+) a = a(a^+ aa^+ a)^T a^+ a = aa^T (a^+)^T a^T (a^+)^T a^+ a \quad (4.1)$$

$$= aa^T aa^T (a^+)^T a^+ a = ba^T aa^T (a^+)^T a^+ a = ba^+ aa^+ aa^+ a = ba^+ a$$

(By using cor 3.6 in [15])

$$a = a(a^+ aa^+ aa^+) a = aa^+ (aa^+ aa^+)^T a = aa^+ (a^+)^T a^T (a^+)^T a^T a$$

$$= aa^+ (a^+)^T a^T aa^T a = aa^+ (a^+)^T a^T aa^T b = aa^+ aa^+ aa^+ b = aa^+ b$$

$$\therefore aa^+ a = aa^+ b = ba^+ a$$

Similarly we obtain $a^+aa = a^+ba = a^+ab; aaa^+ = aba^+ = baa^v;$

$$(II) \Rightarrow (iii) a = aa^+aa^+a = (aa^+a)a^+a = aa^+ba^+a (*)$$

$$a = aa^+aa^+a = aa^+(aa^+a) = aa^+aa^+b (**)$$

$$a = aa^+aa^+a = (aa^+a)a^+a = ba^+aa^+a (***)$$

$$a = aaa = a(aa^+aa^+a)a = (aaa^+)aa^+aa^+ = baa^+aa^+ab (****)$$

From (*),(**),(***) and (****) we get

$$a = aa^+aa^+b = ba^+aa^+a = aa^+ba^+a = ba + ba^+ab$$

$$(III) \Rightarrow (i) a = aa^+aa^+b \Rightarrow aa^+aa^+a = aa^+aa^+b \Rightarrow (aa^+aa^+)^T a = (aa^+aa^+)^T b$$

$$\Rightarrow (a^+)^T a^T (a^+)^T a^T a = (a^+)^T a^T (a^+)^T a^T b \quad (4.2)$$

Pre multiply ' aa^+ ' on Equation (4.2) both sides

$$\Rightarrow aa^T (a^+)^T a^T (a^+)^T a^T a = aa^T (a^+)^T a^T (a^+)^T a^T b \Rightarrow aa^T a = ba^T a$$

$$\text{Take } a = ba^+aa^+a \text{ we get } aa^T (a^+)^T a^T (a^+)^T = ba^T (a^+)^T a^T (a^+)^T \quad (4.3)$$

Post multiply ' $a^T a$ ' on Equation (4.3) both sides

$$aa^T (a^+)^T a^T (a^+)^T a^T a = ba^T (a^+)^T a^T (a^+)^T a^T a \Rightarrow aa^T a = ba^T a$$

$$\therefore aa^T a = aa^T b = ba^T a$$

$$\text{From Equation (4.3) } aa^T (a^+)^T a^T (a^+)^T = ba^T (a^+)^T a^T (a^+)^T$$

Pre multiply ' a ' and post multiply ' a^T ' on both sides

$$aaa^T (a^+)^T a^T (a^+)^T a^T = aba^T (a^+)^T a^T (a^+)^T a^T \Rightarrow aaa^T = aba^T$$

Similarly we get $aaa^T = baa^T$

$$\therefore aaa^T = baa^T = aba^T \text{ and } a^T aa = a^T ab = a^T ba$$

Hence $a \stackrel{T}{\leq} b$

Theorem 3.2: ^[8] Let $a, b \in T$ and a^+, b^+ exists and $a \stackrel{T}{\leq} b$ then $a^T bb^+ bb^+ = b^+ bb^+ ba^T$

Proof: $aa^T a = aa^T b = aa^T bb^+ bb^+ b$

$$(aa^T a)^T = (aa^T bb^+ bb^+ b)^T$$

$$a^T aa^T = b^T (b^+)^T b^T (b^+)^T b^T aa^T$$

$$= b^+ bb^+ b b^T a a^T = b^+ bb^+ b a^T a a^T (\because b^T a a^T = (a a^T b)^T = (a a^T a)^T)$$

$$a^T aa^T aa^+ = b^+ bb^+ ba^T aa^T aa^+$$

$$a^T aa^T (aa^+)^T = b^+ bb^+ ba^T aa^T (aa^+)^T$$

$$a^T aa^T (a^+)^T a^T = b^+ bb^+ ba^T aa^T (a^+)^T a^T$$

$$a^T aa^T aa^T = b^+ bb^+ ba^T aa^T aa^T$$

$$a^T = b^+ bb^+ ba^T \quad (4.4)$$

Similarly $aa^T a = ba^T a \Rightarrow aa^T a = bb^+ bb^+ b a^T a$

$$(aa^T a)^T = (bb^+ bb^+ ba^T a)^T = a^T ab^T (b^+)^T b^T (b^+)^T b^T$$

$$a^T aa^T = (ba^T a)^T bb^+ bb^+ = (aa^T a)^T bb^+ bb^+$$

$$a^T aa^T aa^T = a^T aa^T aa^T bb^+ bb^+$$

$$a^T = a^T bb^+ bb^+ \quad (4.5)$$

From Equations (4.4) and (4.5)

$$a^T = a^T bb^+ bb^+ = b^+ bb^+ ba^T$$

Theorem 3.3: ^[8] For $a, b \in T$ if $a \stackrel{T}{\leq} b$ then $a = ab^+ bb^+ b = bb^+ bb^+ a = ab^+ bb^+ a = bab^+ bb^+ ab$

Proof: we know that $a \stackrel{T}{\leq} b \Leftrightarrow aa^+ a = aa^+ b = ba^+ a$;

$$\Leftrightarrow aaa^+ = aba^+ = baa^+;$$

$$\Leftrightarrow a^+ aa = a^+ ba = a^+ ab;$$

$$\text{By Theorem 3.1, } a = aa^+ aa^+ b = aa^+ aa^+ (bb^+ bb^+ b) = ab^+ bb^+ b \quad (4.6)$$

$$a = ba^+ aa^+ a = (bb^+ bb^+ b) a^+ aa^+ a = bb^+ bb^+ a \quad (4.7)$$

$$a = aa^+ aa^+ b = aa^+ (aa^+ ba^+ a) a^+ b = aa^+ aa^+ (bb^+ bb^+ b) a^+ aa^+ b$$

$$= ab^+ bb^+ b (a^+ aa^+ aa^+) aa^+ b = ab^+ bb^+ ba^+ aa^+ a$$

$$a = ab^+ bb^+ a \quad (4.8)$$

$$a = baa^+ ba^+ ab = ba(a^+ aa^+ aa^+) ba^+ ab = baa^+ aa^+ ab$$

$$= baa^+ (aa^+ ba^+ a) a^+ ab = baa^+ aa^+ (bb^+ bb^+ b) a^+ aa^+ ab$$

$$a = bab^+ bb^+ ab \quad (4.9)$$

From Equations (4.6), (4.7), (4.8), (4.9) we get

$$a = ab^+ bb^+ b = bb^+ bb^+ a = ab^+ bb^+ a = bab^+ bb^+ ab$$

Theorem 3.4: ^[8] Let $a, b \in T$, and $a \stackrel{T}{\leq} b$. If b^+ exists then $b^+ bb^+ \in a\{1,3\} \cap a\{1,4\}$.

Proof: Using Theorem 3.2 and 3.3

$$a = ab^+ bb^+ b = bb^+ bb^+ a \text{ and } a^T = a^T bb^+ bb^+ = b^+ bb^+ ba^T$$

$$a^T = a^T bb^+ bb^+ \Rightarrow a^T aa^T aa^T = a^T aa^T a (a^T bb^+ bb^+)$$

$$a^T = a^T aa^T (aa^T b) b^+ bb^+ = a^T aa^T (aa^T a) b^+ bb^+ = a^T ab^+ bb^+$$

$$(a^T)^T = (a^T ab^+ bb^+)^T$$

$$a = (b^+)^T b^T (b^+)^T a^T a$$

$$\text{Let } ab^+ bb^+ ab^+ bb^+ a = (b^+)^T b^T (b^+)^T a^T a (b^+ bb^+) a (b^+ bb^+) a$$

$$= (b^+)^T b^T (b^+)^T a^T a (b^+ bb^+) a \because a^T = a^T ab^+ bb^+$$

$$= (b^+)^T b^T (b^+)^T a^T a = a$$

$$\therefore ab^+ bb^+ ab^+ bb^+ a = a$$

$$(ab^+ bb^+ e)^T = ((b^+)^T b^T (b^+)^T a^T ab^+ bb^+ e)^T$$

$$= (e(b^+)^T b^T (b^+)^T a^T eab^+ bb^+ e)^T$$

$$= (e^T (b^+)^T b^T (b^+)^T a^T a e^T ab^+ bb^+ e^T)$$

$$=(e(b^+)^T b^T (b^+)^T a^T e a b^+ b b^+ e)$$

$$=(e^T (b^+)^T b^T (b^+)^T a^T a e^T a b^+ b b^+ e^T)$$

$$= a b^+ b b^+ e$$

$$\text{Similarly } a^T = b^+ b b^+ b a^T \Rightarrow a^T = b^+ b b^+ a a^T$$

$$(a^T)^T = (b^+ b b^+ a a^T)^T \Rightarrow a = a a^T (b^+)^T b^T (b^+)^T$$

$$(e b^+ b b^+ a)^T = (e b^+ b b^+ a e a^T (b^+)^T b^T (b^+)^T e)^T$$

$$= e b^+ b b^+ a a^T (b^+)^T b^T (b^+)^T = e b^+ b b^+ a$$

$$\therefore b^+ b b^+ \in a\{1,3\} \cap a\{1,4\}.$$

Corollary 4.1 ^[9] If $a \stackrel{T}{\leq} b$ and a^+ exists then $b \in a^+\{1\}$

$$\text{Proof: } a^+ = a^+ a a^+ a a^+ = a^+ (a a^+ a) a^+ = a^+ (b a^+ a) a^+ = a^+ b a^+ (b a^+ a a^+ a) a^+ \\ = a^+ b a^+ b a^+$$

$$a^+ = a^+ b a^+ b a^+$$

$$\therefore b \in a^+\{1\}.$$

Theorem 3.5 ^[9] Let $a, b \in T$ If a^+ and b^+ exists, then the subsequent are equivalent:

- $a \stackrel{T}{\leq} b$
- $a^+ a a^+ = b^+ a a^+ = a^+ a b^+; a a^+ a^+ = a a^+ b^+ = a b^+ a^+; a^+ a^+ a = a^+ b^+ a = b^+ a^+ a$
- $a^+ = b^+ a a^+ a a^+ = a^+ a a^+ a b^+ = a^+ a b^+ a a^+ = b a^+ a b^+ a a^+ b$

Proof: (i) $\Rightarrow$ (ii)

$$\text{Since } a a^T a = a a^T b$$

$$(a a^T a)^T = (a a^T b)^T = b^T a a^T = b^+ a a^+ (\because b^T = b^+)$$

$$\text{And } a a^T a = b a^T a \Rightarrow (a a^T a)^T = (b a^T a)^T \Rightarrow a^T a a^T = a^T a b^T \Rightarrow a^+ a a^+ = a^+ a b^+$$

$$\therefore a^+ a a^+ = b^+ a a^+ = a^+ a b^+.$$

$$\text{Similarly we get, } a a^+ a^+ = a a^+ b^+ = a b^+ a^+; a^+ a^+ a = a^+ b^+ a = b^+ a^+ a$$

$$(II) \Rightarrow (iii) \quad a^+ a a^+ = b^+ a a^+ \Rightarrow (a^+ a a^+) a a^+ = (b^+ a a^+) a a^+ \Rightarrow a^+ = b^+ a a^+ a a^+$$

$$a^+ a a^+ = a^+ a b^+ \Rightarrow a^+ a (a^+ a a^+) = a^+ a (a^+ a b^+) \Rightarrow a^+ = a^+ a a^+ a b^+$$

$$a^+ = a^+ a a^+ a a^+ = (a^+ a a^+) a a^+ = a^+ a b^+ a a^+ \Rightarrow a^+ = a^+ a b^+ a a^+$$

$$a^+ = a^+ a a^+ a a^+ = a^+ (a^+ a a^+ a a^+) a^+ = (a^+ a^+ a) a^+ a a^+ a^+ = (b^+ a^+ a) a^+ a a^+ a^+ = b^+ a^+ a a^+ (a a^+ a^+)$$

$$= b^+ a^+ a a^+ (a a^+ b^+) = b^+ (a^+ a a^+) a a^+ b^+ = b^+ a^+ a b^+ a a^+ b^+$$

$$\therefore a^+ = b^+ a^+ a b^+ a a^+ b^+$$

$$(III) \Rightarrow (i) \quad a^+ = b^+ a a^+ a a^+ = a^+ a a^+ a b^+$$

$$a^+ = a^+ a a^+ a b^+$$

$$(a^+ a a^+ a a^+)^T = (a^+ a a^+ a b^+)^T \Rightarrow a a^T a a^T a = b a^T a a^T a (\because a^T = a^+)$$

$$\Rightarrow (a a^T a a^T a) a^T a = (b a^T a a^T a) a^T a$$

$$\Rightarrow a a^T a = b a^T a$$

$$\text{Similarly } a^+ = b^+aa^+aa^+ \Rightarrow (a^+aa^+aa^+)^T = (b^+aa^+aa^+)^T$$

$$aa^Taa^Ta = aa^Taa^Tb \Rightarrow aa^T(aa^Taa^Ta) = aa^T(aa^Taa^Tb)$$

$$\Rightarrow aa^Ta = aa^Tb$$

$$\therefore aa^Ta = ba^Ta = aa^Tb (*)$$

$$a^+ = b^+aa^+aa^+ \Rightarrow (a^+aa^+aa^+)^T = (b^+aa^+aa^+)^T \Rightarrow aa^Taa^Ta = aa^Taa^Tb$$

$$\Rightarrow a^T(aa^Taa^Ta) = a^T(aa^Taa^Tb)a \Rightarrow a^Taa = a^Tba$$

$$\text{Similarly we obtain } a^Taa = a^Tab$$

$$\therefore a^Taa = a^Tba = a^Tab (**)$$

$$a^+ = a^+aa^+ab^+ \Rightarrow (a^+aa^+aa^+)^T = (a^+aa^+ab^+)^T \Rightarrow aa^Taa^Ta = ba^Taa^Ta$$

$$\Rightarrow (aa^Taa^Ta)a^T = a(ba^Taa^Ta)a^T \Rightarrow aaa^T = aba^T$$

$$\text{and similarly, } aaa^T = baa^T$$

$$\therefore aaa^T = aba^T = baa^T (***)$$

$$\text{From } (*), (**) \text{ and } (***) \text{ we get } a \leq^T b$$

Theorem 3.6 ^[11] For $a, b \in T^+$, then the following are hold

- $a \leq^T a$
- $a \leq^T b$ and $b \leq^T a$ then $a = b$
- $a \leq^T b$ and $b \leq^T c$ then $a = c$

Proof: (i) $a \leq^T a$ is obvious

$$(ii) a \leq^T b \Leftrightarrow a = aa^+aa^+b = ba^+aa^+a = aa^+ba^+a = baa^+ba^+ab$$

$$b \leq^T a \Leftrightarrow b = ab^+bb^+b = bb^+bb^+a = bb^+ab^+b = abb^+ab^+ba$$

$$a = aa^+aa^+a = ba^+aa^+a = ba^+(aa^+ba^+a)a^+b$$

$$= (bb^+bb^+a)a^+aa^+ba^+aa^+(ab^+bb^+b)$$

$$= bb^+bb^+aa^+aa^+ab^+bb^+b (\because a = ba^+aa^+a)$$

$$= bb^+bb^+(ba^+aa^+a)a^+aa^+(aa^+aa^+b)b^+bb^+b$$

$$= bb^+bb^+ba^+aa^+aa^+aa^+(bb^+bb^+a)b^+bb^+b$$

$$= bb^+bb^+ba^+aa^+(aa^+aa^+b)b^+bb^+ab^+bb^+b$$

$$= bb^+bb^+ba^+aa^+ab^+bb^+ab^+bb^+b$$

$$= bb^+bb^+(ba^+aa^+a)b^+bb^+ab^+bb^+b$$

$$= bb^+bb^+ab^+bb^+ab^+bb^+b$$

$$= bb^+bb^+a(b^+bb^+)ab^+bb^+b = b(b^+bb^+)b = b$$

$$\therefore a = b$$

(III) By Theorem (3.3)

$$a \leq^T b \Leftrightarrow a = ab^+bb^+b = bb^+bb^+a = ab^+bb^+a = bab^+bb^+ab$$

By Theorem (3.1)

$$b \leq^T c \Leftrightarrow b = bb^+bb^+c = cb^+bb^+b = bb^+cb^+b = cbb^+cb^+bc$$

$$a = ba^+aa^+a = (cb^+bb^+b)a^+aa^+a$$

$$\text{By Theorem (3.5(iii)), } a = cb^+bb^+b(b^+aa^+aa^+)aa^+a = c(b^+aa^+aa^+)aa^+a$$

$$a = ca^+aa^+a$$

$$\therefore a \preceq^T c$$

Hence T-ordering $\preceq^T$ is a partial ordering.

Theorem 3.7: ^[14] If $a \preceq^T b$, then we have

- $a^{TT} \preceq^T b^T$
- $a^+ \preceq^T b^+$
- If $b^+ = b^T$ then $a^+ = a^T$
- If $b = b^3$ then $a = a^3$
- If $b = 0$ then $a = 0$

Proof

$$a \preceq^T b \Leftrightarrow aa^T a = aa^T b = ba^T a; a^T aa = a^T ba = a^T ab; aaa^T = aba^T = baa^T$$

$$(aa^T a)^T = (aa^T b)^T = (ba^T a)^T \Rightarrow a^T aa^T = b^T aa^T = a^T ab^T$$

$$\text{Similarly } a^T a^T a = a^T b^T a = b^T a^T a \text{ and } aa^T a^T = aa^T b^T = ab^T a^T$$

$$\therefore a^T \preceq^T b^T$$

$$(II) (aa^T a)^T = (aa^T b)^T = (ba^T a)^T \Rightarrow a^T aa^T = b^T aa^T = a^T ab^T$$

$$\Rightarrow a^+aa^+ = b^+aa^+ = a^+ab^+ (\because a^T = a^+)$$

$$\text{Similarly we get } aa^+a^+ = aa^+b^+ = aa^+b^+ \text{ and } a^+a^+a = b^+a^+a = a^+b^+a$$

$$\therefore a^+ \preceq^T b^+$$

$$(III) a^+ = a^+aa^+aa^+ = b^+aa^+ab^+ = b^+ba^+ab^+$$

$$= b^+(ba^T a)b^T (\because a \preceq^T b \Leftrightarrow a^+aa^+ = b^+aa^+ = a^+ab^+; aa^+a^+ = aa^+b^+ = ab^+a^+; a^+a^+a = a^+b^+a = b^+a^+a)$$

$$= b^+b(a^T ab^T) \text{ using (i)}$$

$$= b^+bb^+aa^T = b^+bb^+(aa^T aa^T a)a^T$$

$$= b^+bb^+(ba^T aa^T a)a^T = b^+bb^+ba^T = a^T$$

$$\therefore a^+ = a^T$$

$$(IV) a^3 = (aa^+aa^+a)a(aa^+aa^+a)$$

$$= aa^+(aa^+a)(aaa^+)aa^+a = aa^+(aa^+b)(baa^+)aa^+a$$

$$= aa^+aa^+bb(aa^+a)a^+a = aa^+aa^+bb(ba^+a)a^+a$$

$$= aa^+aa^+b^3a^+aa^+a = aa^+aa^+ba^+aa^+a (\#)$$

$$= aa^+aa^+a = a$$

$$\therefore a^3 = a$$

$$(V) \text{ From Equation (\#) } a^3 = aa^+aa^+ba^+aa^+a$$

$$= aa^+aa^+(0)a^+aa^+a = 0$$

$$\therefore a^3 = 0.$$

Corollary 3.7: ^[11] If $a \stackrel{T}{\leq} b$ and a^+, b^+ exists then $b^+ \in a\{1\}$

Proof: Since $a^+ = b^+aa^+aa^+ = a^+aa^+ab^+ = a^+abaa^+$

$$\begin{aligned} a &= aa^+aa^+a = a(b^+aa^+aa^+)aa^+a = ab^+a(a^+aa^+)aa^+a \\ &= ab^+ab^+aa^+aa^+a = ab^+ab^+a \end{aligned}$$

$$\therefore a = ab^+ab^+a$$

Hence $b^+ \in a\{1\}$

Theorem 3.8: ^[11] Let $a, b \in T^+$, Then the following conditions are equivalent

- $a \stackrel{T}{\leq} b$
- $b \in a^+\{1, 3, 4\}$

Proof: (i) $\Rightarrow$ (ii)

By corollary (3.1), $b \in a^+\{1\}$

Since $a = aa^+aa^+b = ba^+aa^+a$ and $a^+ = b^+aa^+aa^+ = a^+aa^+ab^+$

$$\begin{aligned} (a^+be)^T &= (b^+aa^+aa^+be)^T = (eb^+aa^+aa^+ebe)^T = e^Tb^Te^Ta^+Ta^Ta^+Ta^Tb^+Te^T \\ &= eb^+aa^+aa^+ebe = a^+be \quad (\because a^+ = a^T \text{ and } e^T = e) \end{aligned}$$

$$\therefore (a^+be)^T = (a^+be)$$

$$(eba^+)^T = (ebeb^+aa^+ab^+e)^T$$

$$= e^Tb^+Ta^Ta^+Ta^Ta^+Te^Tb^+Te^T$$

$$= eba^+aa^+ab^+ = eba^+$$

$$\therefore (eba^+)^T = (eba^+)$$

Hence $b \in a^+\{1, 3, 4\}$.

$$(ii) \Rightarrow (i) \quad a^+ba^+ba^+ = a^+, (a^+be)^T = (a^+be), (eba^+)^T = (eba^+)$$

$$aa^+a = a(a^+ba^+ba^+)a = aa^+eba^+eba^+a$$

$$(aa^+e)(ba^+e)(ba^+e)a = (aa^+e)^T(ba^+e)^T(ba^+e)^T$$

$$= (ba^+eba^+eaa^+e)^Ta = (ba^+ba^+aa^+e)^Ta$$

$$= (ba^+ba^+(ba^+aa^+a)a^+e)^Ta$$

$$= (ba^+eba^+eba^+e)^Ta = (ba^+e)^Ta = ba^+a$$

$$\therefore aa^+a = ba^+a$$

$$aa^+a = a(a^+ba^+ba^+)a = a(a^+be)^T(a^+be)^T(a^+ae)^T$$

$$= a(a^+aea^+bea^+be)^T = a(a^+(aa^+aa^+b)a^+ba^+b)^T$$

$$= a(a^+ba^+ba^+be)^T = a(a^+be)^T = aa^+b$$

$$\therefore aa^+a = ba^+a = aa^+b$$

Similarly we get $aaa^+ = aba^+ = baa^+; a^+aa = a^+ab = a^+ba$

Hence $a \stackrel{T}{\leq} b$

We get the following result by using Theorem (3.1) and Theorem (3.3)

Theorem 3.9: If $a \leq^T b$, then

$$a = aa^+aa^+b = ba^+aa^+a = aa^+ba^+a = baa^+ba^+ab;$$

$$a = ab^+bb^+b = bb^+bb^+a = ab^+bb^+a = bab^+bb^+ab.$$

$$aa^Taa^Tb = ba^Taa^Ta = aa^Tba^Ta = baa^Tba^Tab;$$

$$ab^Tbb^Tb = bb^Tbb^Ta = ab^Tbb^Ta = bab^Tbb^Tab.$$

Conclusion

We proved that T-ordering is a partial ordering. We derived some basic properties of partial T-ordering on Ternary semiring.

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